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Continuous and inverse optimal control designs for chained systems: A global state-scaling transformation and a time-scaling method

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SUMMARY

In this paper, the inverse optimal control designs for chained systems are investigated. The presented designs are based on the thorough study of controllability of chained systems. Particularly, two methods are proposed to recover uniform complete controllability for the chained system. One involves a global singularity-free state-scaling transformation, the other is based on a time transform, and both of them require an innovative design of dynamic control component for its subsystem. Using either of the approaches, the chained system is mapped into a controllable linear time-varying system for which control can systematically be designed to ensure exponential convergence or asymptotic stability. Both state-feedback and output-feedback designs are presented and literally shown to be inversely optimal. Simulation results are used to verify the effectiveness of the proposed controls. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In this paper, we consider the feedback stabilization of the following 2-input and n-dimensional nonholonomic chained system:

$$\dot{x}_1 = u_1, \quad \dot{x}_2 = x_3 u_1, \ldots, \dot{x}_{n-1} = x_n u_1, \quad \dot{x}_n = u_2 \tag{1}$$

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where $x=[x_1 \;\; \dots \;\; x_n]^{\mathrm{T}} \in \Re^n$ is the state, $u=[u_1 \;\; u_2]^{\mathrm{T}} \in \Re^2$ is the control input, and $y=[x_1 \;\; x_2]^{\mathrm{T}} \in \Re^2$ is the output. The feedback stabilization problem of nonholonomic chained systems has been studied by many researchers, and a variety of controls have been proposed, for instance, see a survey [1] and references therein. It is well known that the main difficulty in stabilizing chained system is due to its inherent topological obstructions to the existence of continuous and time-invariant state-feedback controls [2]. In addition, although the chained system is known to be nonlinearly controllable (i.e. small-time controllable [3]), its linearization around the origin is not controllable, nor is its linear time-varying (LTV) subsystem by itself linearly controllable considering the cascaded structures of two subsystems for the chained system due to the vanishing control component $u_1(t)$ (which is required for the stabilization problem). The current existing stabilization controls typically rely upon the use of either discontinuous [4–8] or time-varying feedback [9–11] or both [1]. Other than asymptotic or exponential stabilization, there has been few results on optimal control of nonholonomic systems. New design methodologies to render simpler designs and better controls can be developed by investigating the ways of recovering linear controllability.

In this paper, we concentrate on the problem of investigating controllability and designing controllability-motivated and performance-oriented feedback controls for nonholonomic chained systems. Since many practical nonholonomic systems can be transformed into the chained form by coordinate and input transformations, the proposed control designs based on chained form ensure their wide applicability. Specifically, we attempt to provide positive answers to the following questions: Can the controllability dichotomy aforementioned be reconciled? Is the chained system uniform complete controllable in some time domain (transformed from time t)? Is there a global singularity-free transformation that maps the chained system into a controllable linear system? Are there smooth optimal controls to stabilize the chained system? Using and extending the concept of uniform complete controllability, we show how to construct a global state-scaling transformation and a time-scaling transform to recover linear controllability in the transformed state or time space. In particular, upon presenting a new dynamic control design for component $u_1(t)$, a global nonlinear state transformation is found or a time transform is found to transform the chained system into a controllable LTV system. Accordingly, the well-known Riccati equation and linear optimal control design can then be applied to provide systematic and straightforward solutions to the stabilization problem of the chained system. Both state-feedback and output-feedback controls are presented with explicit illustration of their inverse optimality. The proposed results bridge the gaps between linear and nonlinear controllability, between linear and nonlinear control designs, and between discontinuous stabilization (using state-scaling transformation with singularity) and continuous time-varying stabilization (with singularity-free transformation).

This paper is organized as follows. In Section 2, the control problem for chained systems is formulated with motivation and background for the present study. Section 3 gives the state-feedback control using state-scaling method and Section 4 presents the time-scaling method-based state-feedback control. In Section 5, the output-feedback controls based on the proposed two design methods are presented. Simulation results are provided in Section 6, and Section 7 concludes the paper. The related discussions on uniform complete controllability of LTV-systems are collected in Appendix A.

2. PROBLEM FORMULATION

The objective of this paper is to present two design methods of time-varying and continuous feedback controls that globally stabilize system (1) and are optimal with respect to certain performance

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indices. It is straightforward to extend the proposed results to m-input nonholonomic systems that can be transformed into the chained form.

2.1. Motivation and background

Chained system (1) can be decomposed into the following two interconnected subsystems:

$$\dot{x}_1 = u_1 \tag{2}$$

and

$$\dot{z} = u_1 A_2^* z + B_2 u_2 \tag{3}$$

where $z = [z_1 \ \ z_2 \ \ \ldots \ \ z_{n-1}]^{\mathrm{T}} \triangleq [x_2 \ \ x_3 \ \ \ldots \ \ x_n]^{\mathrm{T}}$, and

$$A_2^* \triangleq \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & & 1 \\ 0 & 0 & \cdots & & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{4}$$

As such, it is well known that the chained system has several nice properties: (i) subsystem (2) is linear, and u_1 can easily be designed to stabilize x_1; (ii) subsystem (3) is a chain of u_1-weighted integrators; and (iii) system (1) is nonlinearly controllable everywhere as Lie brackets on its vector fields are of full rank.

On the other hand, stabilization of chained system (1) remains to be a difficult and interesting problem because of the following technical issues: (i) Topologically, the chained system cannot be stabilized under any continuous control $u = u(x)$ due to its nonlinear characteristics [2]; (ii) While the system is nonlinearly controllable everywhere, the system is not globally feedback linearizable (although local feedback linearizable is possible as shown by the σ-process but singularity manifold remains in all the neighborhoods around the origin), and nonlinear controllability does not necessarily translate into systematic control design; and (iii) system (1) is not linearly controllable around the origin. The apparent dichotomy between nonlinear and linear controllability properties is of particular importance as it characterizes both difficulty of control design and need of having systematic design and improving control performance.

In contrast to the stabilization problem, the tracking problem is generally easier to be solved for the chained system. This is because tracking control component u_1 designed for subsystem (2) is often nonvanishing, in which case subsystem (3) is uniformly completely controllable. Hence, the tracking control can be systematically designed not only to ensure exponential stability for the tracking error but also to achieve near-optimal performance (the best achievable real time) [12]. In the stabilization problem, subsystem (3) is never uniformly completely controllable. Nonetheless, it is shown in this paper that either state- or time-scaling transformation can be used to recover uniform complete controllability for subsystem (3). By removing the controllability dichotomy, we are able to systematically design continuous controls to globally stabilize the chained system and to achieve optimal performance with respect to certain performance index. In the following subsections, the state- and time-scaling transformations are motivated.

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2.2. *Existing transformations and their singularity*

Among the approaches used to stabilize nonholonomic chained systems, discontinuous control designs are more straightforward. This is because discontinuous stabilizing control $u=u(x)$ does exist and typically consists of two switching laws. One of the laws is to stabilize the system in a nonsingular subspace by applying a state transformation and mapping the system into a linear system, and the other is to bring the system to the nonsingular subspace. Such a discontinuous design overcomes the loss of linear controllability by switching, and it avoids the difficulty of designing a single continuous but time-varying control.

The so-called σ-process proposed by Astolfi [4] is the common representative of existing discontinuous designs, and its nonsingular subspace is defined by $\Omega=\{x\in\Re^n : x_1\neq 0\}$. If $x(t_0)\notin\Omega$, a finite-time control law of $u_1(t)$ being a constant can be selected under which $x_1(t_0+\Delta t)\neq 0$ for some $\Delta t>0$. If $x(t_0)\in\Omega$ or once $x_1(t_0+\Delta t)\neq 0$ is accomplished, control component $u_1=-k_1x_1$ with $k_1>0$ is stabilizing for subsystem (2), the following state-scaling transformation:

$$\xi_i=\frac{z_i}{x_1^{n-i}}, \quad 1\leqslant i\leqslant n-1 \tag{5}$$

can be applied to subsystem (3) to render the linear time-invariant (LTI) controllable system

$$\dot{\xi}_i=(n-i)k_1\xi_i-k_1\xi_{i+1}, \quad i=1,\ldots,n-2$$

and

$$\dot{\xi}_{n-1}=k_1\xi_{n-1}-k_1u_2$$

and control component u_2 can easily be found to stabilize the above LTI system.

Clearly, a control design based on the σ-process is quite simple. Nonetheless, such a design has three shortcomings. First, the resulting control is discontinuous by nature. Second, the state-scaling transformation is well defined everywhere except on the hyperplane of $x_1=0$; although a separate control law keeps the state off this singularity hyperplane, the transformation and the resulting control contain such terms as $\xi_i(t)=x_{i+1}(t)/x_1^{n-i}(t)$, which may assume excessively large values in the neighborhood around the singularity hyperplane and during the transient. Finally, the calculation of $\xi_i(t)=x_{i+1}(t)/x_1^{n-i}(t)$ becomes numerically problematic as both $x_i(t)$ and $x_1(t)$ are supposed to approach zero exponentially but their measurements may contain noises.

2.3. *State-scaling and time-scaling methods*

One of the topics studied in this paper is whether the σ-process can be improved so that a single time-varying continuous control law can be designed in a systematic and straightforward manner to stabilize the class of nonholonomic systems. A positive answer is proposed in the paper by proposing a new dynamic-feedback control $u_1(t)$ and a new singularity-free state-scaling transformation. Under the proposed global state-scaling transformation, subsystem (3) is mapped into a LTV system for which uniform complete controllability is established and a time-varying continuous control is designed which makes the system states converge to the origin exponentially. The proposed new approach not only overcomes the first two shortcomings but also renders an optimal control.

On the other hand, to recover linear controllability of subsystem (3), a time-scaling method is also used to design a continuous asymptotically stabilizing control. The time-scaling method

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requires a transform of time but not any state transformation. This alternative design overcomes all the three shortcomings aforementioned, and it renders an optimal control as well.

3. GLOBAL STATE-SCALING TRANSFORMATION AND DESIGN OF CONTINUOUS EXPONENTIALLY CONVERGENT OPTIMAL CONTROL

In this section, a dynamic-feedback control design of component u_1 is proposed. Based on the design, a global state-scaling transformation is introduced to overcome the singularity problem of the existing scaling transformations. This new transformation enables the designer to recover uniform complete controllability for the chained system and to design a class of continuous, time-varying and optimal controls that make the system states converge to the origin exponentially.

3.1. Dynamic control component u_1

The proposed control for component $u_1(t)$ is

$$\dot{u}_1 = -(k_1+\zeta)u_1 - k_1\zeta x_1, \quad u_1(t_0) = -k_1 x_1(t_0) + \|x(t_0)\| \tag{6}$$

where $k_1>0$ is a feedback control gain, and constant $0<\zeta<k_1$ is a design parameter arbitrarily chosen.

It follows from (2) that, under (6), the closed-loop system of state variable x_1 becomes

$$\ddot{x}_1 + (k_1+\zeta)\dot{x}_1 + k_1\zeta x_1 = 0$$

Therefore, the closed-loop solution is

$$x_1(t) = c_1 \mathrm{e}^{-k_1(t-t_0)} + c_2 \mathrm{e}^{-\zeta(t-t_0)}$$

and

$$u_1(t) = -k_1 c_1 \mathrm{e}^{-k_1(t-t_0)} - \zeta c_2 \mathrm{e}^{-\zeta(t-t_0)} \tag{7}$$

where $c_1(x_0) \triangleq x_1(t_0) - c_2$ and $c_2(x_0) \triangleq \|x(t_0)\|/(k_1-\zeta)$. Clearly, through injecting $u_1(t_0)$, control (6) is the simplest controller to exciting x_1 whenever $\|x(t_0)\| \neq 0$ while making both u_1 and x_1 exponentially convergent (and asymptotically stable with respect not to $|x_1(t_0)|$ but $\|x(t_0)\|$).

Since feedback control u_1 in (6) is dynamic, u_1 is now an internal state variable. Accordingly, for the system consisting of (2) and (6), we can define the following 'output':

$$y_{1d} \triangleq \frac{1}{k_1-\zeta}[k_1 x_1 + u_1] \tag{8}$$

It follows from (2) and (6) that

$$\dot{y}_{1d} = -\zeta y_{1d} \tag{9}$$

where $y_{1d}(t_0) = \|x(t_0)\|/(k_1-\zeta) \neq 0$ if $\|x(t_0)\| \neq 0$. Hence, unless $\|x(t_0)\| = 0$, $y_{1d} \neq 0$ for all $t \in [t_0, \infty)$, which makes it possible to find a global scaling transformation.

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3.2. A global state-scaling transformation

Since y_{1d} in (8) has been shown to be nonzero for all finite t, we can propose the following scaling transformation that is differentiable and never singular: with $\xi = [\xi_1 \;\; \dots \;\; \xi_{n-1}]^{\mathrm{T}}$ and for $i = 1, \dots, (n-1)$

$$\xi_i = \begin{cases} 0 & \text{if } \|x(t_0)\| = 0 \\ \dfrac{z_i}{y_{1d}^{n-1-i}} = \dfrac{z_i(k_1 - \zeta)^{n-1-i}}{(u_1 + k_1 x_1)^{n-1-i}} & \text{if else} \end{cases} \tag{10}$$

It follows from (9) that, under transformation (10), subsystem (3) is mapped into

$$\dot{\xi} = \begin{cases} 0 & \text{if } \|x(t_0)\| = 0 \\ F_2(t)\xi + B_2 u_2 & \text{if else} \end{cases} \tag{11}$$

where

$$F_2(t) \triangleq \mathrm{diag}\{\zeta(n-2), \dots, \zeta, 0\} + \alpha(t, x_0) A_2^* \tag{12}$$

with

$$\alpha(t, x_0) \triangleq \frac{u_1(t)}{y_{1d}(t)} = c_3 \mathrm{e}^{-(k_1 - \zeta)(t - t_0)} - \zeta \tag{13}$$

for

$$c_3(x_0) \triangleq -k_1 \left[\frac{(k_1 - \zeta) x_1(t_0)}{\|x(t_0)\|} - 1 \right]$$

It follows from (7) and (9) that, so long as $\|x(t_0)\| \neq 0$,

$$\lim_{t \to \infty} \alpha(t, x_0) = -\zeta < 0$$

which shows that $\alpha(t, x_0)$ is uniformly nonvanishing (see the definition and discussions in Appendix A). Using this property, design of u_2 can be proceeded with to ensure global exponential stability of the overall system.

Remark 3.1
It should be noted that control (6) is conventional, purely feedback and dynamic. Control (6) has the key feature that, if $x_1(t_0) = 0$ but $\|x(t_0)\| \neq 0$, it moves $x_1(t)$ by injecting $\|x(t_0)\|$ as the initial condition of $u_1(t_0)$ while ensuring the convergence of $\|x(t)\|$ through the control design of u_2.

3.3. Dynamic control component u_2

Let dynamic control component u_2 be defined by

$$u_2(t) = -r_2^{-1}(t) B_2^{\mathrm{T}} P_2(t) \xi \tag{14}$$

where $\underline{r}_2 \leqslant r_2(t) \leqslant \overline{r}_2$ for some positive constants $\underline{r}_2$ and $\overline{r}_2$, and $P_2(t)$ is the solution to Riccati equation (A4) (see Appendix A) with $C_2 = [1 \;\; 0 \;\; \dots \;\; 0] \in \Re^{1 \times (n-1)}$ and with matrix $F_2(t)$ defined from (12) to (13). The closed-loop convergence under control (6) and (14) is stated as the following theorem.

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Theorem 3.1
Under dynamic-feedback control (6) and (14), system (1) is globally exponentially convergent with bounded input.

Proof
It is clear from (6) and (10) that $\|x(t_0)\|=0$ implies $u_1=u_2=0$ and hence $x(t)\equiv 0$.

Now consider the case that $\|x(t_0)\|\neq 0$. Although Lemma A.2 in Appendix A is not applicable to transformed subsystem (11), uniform complete controllability of pairs $\{F_2(t), B_2\}$ and $\{F_2^{\mathrm{T}}(t), C_2^{\mathrm{T}}\}$ can be shown by noting that $F_2(\infty)=\lim_{t\to\infty} F_2(t)$ exists and that $\{F_2(\infty), B_2\}$ and $\{F_2^{\mathrm{T}}(\infty), C_2^{\mathrm{T}}\}$ are constant and controllable.

Let Lyapunov function be

$$V(x_1,u_1,\xi)=\tfrac{1}{2}[u_1+k_1x_1]^2+\tfrac{1}{2}[u_1+\zeta x_1]^2+\xi^{\mathrm{T}}P_2(t)\xi$$

It follows from (2), (6), and (11) that

$$\dot{V}(x_1,u_1,\xi)=-\zeta[u_1+k_1x_1]^2-k_1[u_1+\zeta x_1]^2-\xi^{\mathrm{T}}[C_2^{\mathrm{T}}q_2C_2+P_2B_2r_2^{-1}B_2^{\mathrm{T}}P_2]\xi$$

from which exponential stability of x_1, u_1, and ξ can easily be concluded using Lemma A.1 in Appendix A. It follows from (9) and (10) that exponential stability of ξ implies that states z converge to the origin exponentially. From (14), the control also exponentially converge to zero and is bounded. □

Remark 3.2
It should be noted that control (14) is also conventional, purely feedback and dynamic. Owing to the facts that the overall system is nonlinear and that optimal performance is sought, gain matrix $P_2(t)$ (the solution to Riccati equation (A4)) depends upon both initial condition $x(t_0)$ and time t (as does vector $F_2(t)$). Nonetheless, function $F(t)$ becomes constant and independent of $x(t_0)$ in the limit of $t\to\infty$, hence differential Riccati equation (A4) approaches the algebraic Riccati equation in the limit, and the steady-state solution $P_2(t)$ can easily be found.

3.4. *Optimal performance*

To quantify performance of the proposed control, let us introduce performance index $J=J_1+J_2$ where

$$J_1=\int_{t_0}^{\infty}\frac{q_1}{(k_1-\zeta)^2}(\zeta x_1+u_1)^2\,\mathrm{d}t \tag{15}$$

and

$$J_2=\frac{1}{2}\int_{t_0}^{\infty}\left[\frac{q_2(t)(k_1-\zeta)^{2n-4}x_2^2}{(u_1+k_1x_1)^{2n-4}}+r_2(t)u_2^2\right]\mathrm{d}t \tag{16}$$

for any positive constant q_1 (and $q_1=1$ can be set without loss of any generality), and positively valued and uniformly bounded time functions $q_2(t)$ and $r_2(t)$.

Theorem 3.2
For system (1), dynamic-feedback control (6) and (14) is optimal with respect to performance index $J=J_1+J_2$, where J_i are defined by (15) and (16).

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Proof
Let us define the following auxiliary variable:

$$\eta_1(t) \triangleq -\frac{1}{k_1-\zeta}[\zeta x_1+u_1] \tag{17}$$

It follows from (2) and (6) that

$$\dot{\eta}_1 = -k_1\eta_1 \triangleq v_1 \tag{18}$$

Now, considering performance index

$$J_1 = \frac{1}{2}\int_{t_0}^{\infty}(q_1\eta_1^2+r_1v_1^2)\,\mathrm{d}t \tag{19}$$

for some constants $r_1, q_1>0$, we know that fictitious control v_1 in (18) is optimal with respect to (19) provided that $k_1 = p_1/r_1$ where $p_1 = \sqrt{q_1 r_1}$. It is straightforward to verify that differential equations (9) and (18) are equivalent to (2) and (6). Hence, control (6) is optimal for system (2), and the corresponding performance index (19) can then be expressed as (15) in terms of the original variables by noting the choice of k_1 and p_1.

Recalling the property of uniform complete controllability revealed in the proof of Theorem 3.1, we know that control (14) optimally stabilizes system (11) under the following performance index:

$$J_2 = \frac{1}{2}\int_{t_0}^{\infty}[\xi^{\mathrm{T}}C_2^{\mathrm{T}}q_2C_2\xi+r_2u_2^2]\,\mathrm{d}t$$

which can be expressed as (16) in terms of the original variables. □

Remark 3.3
The meaning of performance index (15) or (19) can be further explained as follows. Solving u_1 from (8) and substituting the solution into (17) yields $\eta_1 = x_1 - y_{1d}$. On the other hand, differential equation (9) can be rewritten as $\dot{y}_{1d} = v_{1d}$ with $v_{1d} \triangleq -\zeta y_{1d}$. It follows from (17) and (8) and from the definitions of v_1 in (18) and v_{1d} that $u_1 = v_1 + v_{1d}$. Thus, y_{1d} and η_1 can be viewed as the 'desired asymptotically convergent trajectory' and 'tracking error' for $x_1(t)$, respectively; v_{1d} and v_1 can be viewed as 'feedforward control' and 'incremental feedback control,' respectively; and dynamical control u_1 is optimal under (19) to minimize the 'tracking error.' It is also worth mentioning that although performance indices (15) and (16) (or (28) and (29) in Section 4) are quantified on the states and inputs of chained systems, and their physical meaning can be pursued based on the inverse transformations from the chained systems to the original practical nonholonomic systems case by case.

3.5. Computational issue

Mapping (10) is new because it is globally well defined for any initial condition of $x(t_0)$. Nonetheless, it remains to be a state-scaling transformation and hence has a computational shortcoming as do the existing transformations such as the σ-process in (5). Specifically, although both z and y_{1d} are exponentially convergent, implementation of control (14) with transformation (10) calls for computing the ratio of two infinitesimals. Such a computation is numerically unstable, which

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is unavoidable in order to achieve exponential convergence but can be avoided for asymptotic stabilization. In the following section, a new method of time scaling is introduced to overcome the numerical problem.

4. TIME-SCALING METHOD AND DESIGN OF CONTINUOUS ASYMPTOTICALLY STABILIZING OPTIMAL CONTROL

To overcome the computational issue described in Section 3.5, we propose a time-scaling method by which control can be designed to asymptotically stabilizing system (1) without using any state transformation.

4.1. Dynamic control design

In this subsection, two dynamic-feedback control components u_1 and u_2 will sequentially be designed to form the proposed asymptotically stabilizing control. As the first step, dynamic-feedback control u_1 is chosen to be of the following form:

$$\dot{u}_1 = -2\lambda(t)u_1 - \omega^2 x_1, \quad \lambda(t) \triangleq \frac{1}{t - t_0 + 1} \tag{20}$$

where $u_1(t_0) = c_u \|x(t_0)\|$, $\omega > 0$ is a design parameter whose value is arbitrary, c_u is also a design parameter arbitrarily chosen by the designer so long as $c_u \neq 0$ whenever $x_1(0) = 0$. It is apparent that $u_1 \equiv 0$ if $\|x(t_0)\| = 0$. The following lemma provides the property of subsystem (2) under dynamic control (20).

Lemma 4.1
Under dynamic control (20), state x_1 of subsystem (2) is uniformly bounded by $(|c_u| + \omega + 1)\|x(t_0)\|/\omega$ and is also uniformly asymptotically convergent to zero.

Proof
It follows from (20) and (2) that the closed-loop subsystem is

$$\ddot{x}_1 + 2\lambda(t)\dot{x}_1 + \omega^2 x_1 = 0 \tag{21}$$

Since the time function in the above differential equation is continuous and uniformly bounded, the closed-loop subsystem has a unique solution under any given initial conditions of $x_1(t_0)$ and $\dot{x}_1(t_0) = u_1(t_0)$.

It is straightforward to show by direct computations that equations (2), (20), and (21) all hold under the following unique solution:

$$x_1(t) = \lambda(t)\beta(t, x_0), \quad u_1(t) = \lambda(t)\gamma(t, x_0) \tag{22}$$

where

$$\beta(t, x_0) \triangleq x_1(t_0)\cos(\omega t - \omega t_0) + \frac{c_u\|x(t_0)\| + x_1(t_0)}{\omega}\sin(\omega t - \omega t_0)$$

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and

$$\gamma(t,x_0) \triangleq \{c_u\|x(t_0)\| + [1-\lambda(t)]x_1(t_0)\}\cos(\omega t - \omega t_0)$$
$$- \left\{\lambda(t)\frac{c_u\|x(t_0)\|}{\omega} + \left[\frac{\lambda(t)}{\omega} + \omega\right]x_1(t_0)\right\} \times \sin(\omega t - \omega t_0) \tag{23}$$

Hence, uniform boundness and uniform asymptotic convergence are apparent. □

Let dynamic control u_2 for subsystem (3) be

$$u_2(t) \triangleq -\lambda(t) r_2^{-1}(t) B_2^{\mathrm{T}} P_2'(t) z \tag{24}$$

where $\lambda(t)$ is that in (20), $\gamma(t,x_0)$ is the continuous and uniformly bounded time function defined in (23), $0<\underline{r}_2\leqslant r_2(t)\leqslant\overline{r}_2$, $0<\underline{q}_2\leqslant q_2(t)\leqslant\overline{q}_2$, $C_2=[1\ \ 0\ \ \ldots\ \ 0]\in\Re^{1\times(n-1)}$, $r_2'(\tau)\triangleq r_2(e^\tau+t_0-1)$, $q_2'(\tau)\triangleq q_2(e^\tau+t_0-1)$, $\gamma'(\tau,x_0)\triangleq\gamma(e^\tau+t_0-1,x_0)$, $P_2'(t)=P_2'(e^\tau+t_0-1)\triangleq P_2''(\tau)$, and $P_2''(\tau)$ with $\tau\in[0,\infty)$ is the positive-definite solution to the following Riccati equation:

$$0 = \frac{\mathrm{d}P_2''(\tau)}{\mathrm{d}\tau} + \gamma'(\tau,x_0)P_2''(\tau)A_2^* + (A_2^*)^{\mathrm{T}}P_2''(t)\gamma'(\tau,x_0)$$
$$-\frac{1}{r_2'(\tau)}P_2''(\tau)B_2B_2^{\mathrm{T}}P_2''(\tau) + C_2^{\mathrm{T}}q_2'(\tau)C_2 \tag{25}$$

The following theorem describes the result of asymptotic stability.

Theorem 4.1

Under dynamic-feedback control (20) and (24), system (1) is globally asymptotically stable.

Proof

It is clear from (20) and (24) that $\|x(t_0)\|=0$ implies $u_1=u_2=0$ and hence $x(t)\equiv 0$. Next, let us consider the case that $\|x(t_0)\|\neq 0$ and introduce Lyapunov function

$$V'(x_1,u_1,z,t) \triangleq V_1'(x_1,u_1,t) + V_2'(z,t)$$

where

$$V_1'(x_1,u_1,t) \triangleq \tfrac{1}{2}[u_1+\lambda(t)x_1]^2 + \tfrac{1}{2}\omega^2x_1^2$$

and

$$V_2'(z,t) \triangleq z^{\mathrm{T}}P_2'(t)z$$

It follows from (20) and (2) that

$$\frac{\mathrm{d}V_1'}{\mathrm{d}t} = -\lambda(t)[u_1+\lambda(t)x_1]^2 - \lambda(t)\omega^2x_1^2 = -2\lambda(t)V_1'$$

On the other hand, it follows from (3) and (22) that, letting $\tau=\ln(t-t_0+1)$

$$\frac{\mathrm{d}z'}{\mathrm{d}\tau} = \gamma'(\tau,x_0)A_2^*z' + B_2u_2' \tag{26}$$

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where $z'(\tau)=z(t)$ and $u_2'(\tau)=(t-t_0+1)u_2(t)$. System (26) is LTV; by Lemma A.2 in Appendix A, its pair $\{\gamma'(\tau,x_0)A_2^*, B_2\}$ is uniformly completely controllable (in the time domain of both t and τ), and hence Riccati equation (25) has a positive-definite solution $P_2''(\tau)$. Recalling $P_2''(\tau)=P_2'(t)$, we can rewrite (24) as

$$u_2'(\tau)=-\frac{1}{r_2'(\tau)}B_2P_2''(\tau)z' \tag{27}$$

Therefore, we know from the above expression and (26) that

$$V_2'(z,t)\triangleq z^{\mathrm{T}}P_2'(t)z=[z']^{\mathrm{T}}P_2''(\tau)z'\triangleq V_2''(z',\tau)$$

and

$$\begin{aligned}\frac{\mathrm{d}V_2'(z,t)}{\mathrm{d}t}&=\frac{\mathrm{d}V_2'(z,t)}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t}=\lambda(t)\frac{\mathrm{d}V_2''(z',\tau)}{\mathrm{d}\tau}\\&=-\lambda(t)(z')^{\mathrm{T}}\left[C_2^{\mathrm{T}}q_2'C_2+\frac{1}{r_2'}P_2''B_2B_2^{\mathrm{T}}P_2''\right]z'\\&=-\lambda(t)z^{\mathrm{T}}\left[C_2^{\mathrm{T}}q_2C_2+\frac{1}{r_2}P_2'B_2B_2^{\mathrm{T}}P_2'\right]z\end{aligned}$$

Combining the expressions of $\dot V_1'$ and $\dot V_2'$, we have

$$\frac{\mathrm{d}V'(x_1,u_1,z,t)}{\mathrm{d}t}=-2\lambda(t)V_1'-z^{\mathrm{T}}\left[C_2^{\mathrm{T}}q_2C_2+\frac{1}{r_2}P_2'B_2B_2^{\mathrm{T}}P_2'\right]z\leqslant 0$$

from which asymptotic stability of x_1, u_1, and z can be concluded by applying Lemma A.1 in Appendix A. □

Remark 4.1
The proof of Theorem 4.1 shows that both $x(t)=x(e^\tau+t_0-1)\triangleq x'(\tau)$ and $u(t)=u(e^\tau+t_0-1)\triangleq u'(\tau)$ are exponentially stable with respect to τ and that $x(t)$ and $u(t)$ are asymptotically stable. Control design and asymptotic stabilization are accomplished systematically and simply because the proposed time-scaling method recovers uniform complete controllability.

Remark 4.2
Controls (20) and (24) have the properties similar to those of controls (6) and (14) and explained in Remarks 3.1 and 3.2. One minor difference is that, since $\gamma(t,x_0)$ becomes periodic in the limit of $t\to\infty$, matrix $P_2'(t)$ as the solution to Riccati equation (25) also becomes periodic in the limit and hence can easily be found using well-established methods [13].

4.2. *Optimal performance*

The following theorem quantifies optimality of dynamic control (20) and (24).

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Theorem 4.2

For system (1), dynamic-feedback control (20) and (24) is optimal with respect to performance index $J' = J_1' + J_2'$, where

$$J_1' = \frac{1}{2}\int_{t_0}^{\infty} \begin{bmatrix} x_1 \\ u_1 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \lambda^3(t)+\omega^2\lambda(t) & \lambda^2(t) \\ \lambda^2(t) & \lambda(t) \end{bmatrix} \begin{bmatrix} x_1 \\ u_1 \end{bmatrix} \mathrm{d}t \tag{28}$$

and

$$J_2' = \frac{1}{2}\int_{t_0}^{\infty} \left[\lambda(t) q_2(t) x_2^2 + \frac{r_2(t)}{\lambda(t)} u_2^2 \right] \mathrm{d}t \tag{29}$$

for $0 < \underline{r}_2 \leqslant r_2(t) \leqslant \overline{r}_2$ and $0 < \underline{q}_2 \leqslant q_2(t) \leqslant \overline{q}_2$.

Proof

To show optimality of control (20), let us consider the following time-varying linear system:

$$\dot{\overline{x}}_1 = A_1(t)\overline{x}_1 + B_1 v_1 \tag{30}$$

where $\overline{x}_1 \triangleq [x_1 \;\; u_1]^{\mathrm{T}}$, $B_1 = [0 \;\; 1]^{\mathrm{T}}$, and

$$A_1(t) = \begin{bmatrix} 0 & 1 \\ -\omega^2 + \lambda^2(t) & -\lambda(t) \end{bmatrix}$$

Consider performance index

$$J_1' = \frac{1}{2}\int_{t_0}^{\infty} [\overline{x}_1^{\mathrm{T}} Q_1(t) \overline{x}_1 + r_1(t) v_1^2] \, \mathrm{d}t \tag{31}$$

where $r_1(t) = \lambda^{-1}(t)$ and

$$Q_1(t) = \begin{bmatrix} \lambda^3(t) + 2\lambda(t)\omega^2 & \lambda^2(t) \\ \lambda^2(t) & \lambda(t) \end{bmatrix}$$

It is straightforward to show that $Q_1(t)$ is positive semi-definite for $t \geqslant t_0$ and that matrix

$$P_1(t) = \begin{bmatrix} \omega^2 + \lambda^2(t) & \lambda(t) \\ \lambda(t) & 1 \end{bmatrix}$$

is also positive definite and satisfies the Riccati equation

$$0 = \dot{P}_1(t) + P_1(t)A_1 + A_1^{\mathrm{T}} P_1(t) - \frac{1}{r_1(t)} P_1(t) B_1 B_1^{\mathrm{T}} P_1(t) + Q_1(t)$$

Hence, the optimal control for system (30) is $v_1 = v_1^*$, where

$$v_1^* = -\frac{1}{r_1(t)} B_1^{\mathrm{T}} P_1(t) \overline{x}_1 = -\lambda^2(t) x_1 - \lambda(t) u_1 \tag{32}$$

It is simple to show that, under control (32), performance index (31) is identical to that in (28) and system (30) is identical to system (2) under dynamic control (20). Therefore, control (20) is shown to be optimal with respect to performance index (28).

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To show optimality of control (24), consider performance index

$$J_2' = \frac{1}{2}\int_0^{\infty} \{[z'(\tau)]^{\mathrm{T}} C_2 q_2'(\tau) C_2 z'(\tau) + r_2'(\tau)[u_2'(\tau)]^2\}\,\mathrm{d}\tau \tag{33}$$

It follows from Riccati equation (25) that control (27) is optimal for system (26). Applying the time transformation $\tau = \ln(t - t_0 + 1)$ to (33) yields performance index (29). □

5. OUTPUT-FEEDBACK CONTROL DESIGN

In this section, output-feedback control designs are presented as the extensions to the state-feedback results in Sections 3 and 4.

5.1. State-scaling method

Recall that under the state-scaling transformation (10) and the dynamic control (6), subsystem (3) is mapped into (11), which can be further expressed as

$$\dot{\xi} = F_2'\xi + B_2 u_2 + A_2^* \xi \alpha'(t, x_0) \tag{34}$$

where $\alpha'(t, x_0) = \alpha(t, x_0) + \zeta$, and

$$F_2' \triangleq \mathrm{diag}\{\zeta(n-2), \ldots, \zeta, 0\} - \zeta A_2^*$$

Apparently, the LTI pair $\{F_2', C_2\}$ is observable and $\{F_2', B_2\}$ is controllable. Upon the availability of input–output information of x_1, x_2, u_1, and u_2: for any initial condition $\hat{\xi}(t_0)$, we can design the following observer:

$$\dot{\hat{\xi}} = F_2'\hat{\xi} + B_2 u_2 + A_2^* \hat{\xi} \alpha' + H(\xi_1 - \hat{\xi}_1) \tag{35}$$

where $H = [h_1, \ldots, h_{n-1}]^{\mathrm{T}}$ is chosen such that matrix $F_2' - HC_2$ is Hurwitz. It follows from (34) and (35) that dynamics of estimation error $\tilde{\xi} \triangleq \xi - \hat{\xi}$ is given by

$$\dot{\tilde{\xi}} = [F_2' - HC_2 + A_2^* \alpha']\tilde{\xi} \tag{36}$$

Lemma 5.1

Under the choice of $H = [h_1, \ldots, h_{n-1}]^{\mathrm{T}}$ such that matrix $F_2' - HC_2$ is Hurwitz, the origin of dynamics of estimation error (36) is exponentially stable. In addition, there exists a symmetric positive-definite matrix $\tilde{P}_2(t)$ satisfying the matrix differential equation

$$\dot{\tilde{P}}_2(t) = -C_2^{\mathrm{T}} C_2 - [F_2' - HC_2 + A_2^* \alpha']^{\mathrm{T}} \tilde{P}_2(t) - \tilde{P}_2(t)[F_2' - HC_2 + A_2^* \alpha'] \tag{37}$$

Proof

Note that matrix $F_2' - HC_2$ is an asymptotically stable matrix, and

$$\int_{t_0}^{\infty} \|A_2^* \alpha'(t)\|^2\,\mathrm{d}t \leqslant \|A_2^*\|^2 c_3^2 \int_{t_0}^{\infty} \mathrm{e}^{-2(k_1-\zeta)(t-t_0)}\,\mathrm{d}t = \frac{\|A_2^*\|^2 c_3^2}{2(k_1 - \zeta)}$$

Thus, the exponential stability of (36) can be readily concluded by invoking Lemma 2.2 of [14].

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On the other hand, by Lyapunov converse theorem, the existence of Lyapunov function $\tilde{\xi}^{\mathrm{T}}\tilde{P}_2(t)\tilde{\xi}$ satisfying (37) is ensured. □

Using the proposed state observer (35), the corresponding observer-based dynamic control component u_2 becomes

$$u_2(t) = -r_2^{-1}(t)B_2^{\mathrm{T}}\hat{P}_2(t)\hat{\xi} \tag{38}$$

where $\hat{P}_2(t)$ is the positive-definite solution to the following Riccati equation: for any bounded positive-definite matrix $Q_2(t)$

$$0 = \dot{\hat{P}}_2(t) + \hat{P}_2(t)F_2(t) + F_2^{\mathrm{T}}(t)\hat{P}_2(t) + Q_2(t) - \hat{P}_2(t)B_2r_2^{-1}(t)B_2^{\mathrm{T}}P_2(t) \tag{39}$$

Theorem 5.1
Under output-feedback dynamic control (6) and (38), the states of system (1) are globally exponentially convergent and all the closed-loop system signals are bounded. In addition, dynamic-feedback control (6) and (38) are optimal with respect to performance index $\hat{J} = J_1 + \hat{J}_2$, where J_1 is defined by (15) and $\hat{J}_2$ is given by

$$\hat{J}_2 = \int_{t_0}^{\infty}\left\{[\hat{\xi}^{\mathrm{T}} \;\; \tilde{\xi}^{\mathrm{T}}]\hat{Q}_2(t)\begin{bmatrix}\hat{\xi}\\ \tilde{\xi}\end{bmatrix} + u_2^{\mathrm{T}}R_2u_2\right\}\mathrm{d}t \tag{40}$$

where

$$\hat{Q}_2(t) = \begin{bmatrix} Q_2 & -\hat{P}_2HC_2 \\ -C_2^{\mathrm{T}}H^{\mathrm{T}}\hat{P}_2 & \gamma C_2^{\mathrm{T}}C_2 \end{bmatrix}$$

and $\hat{\gamma}>0$ is chosen to satisfy

$$\hat{\gamma}^2Q_2 > \hat{P}_2HH^{\mathrm{T}}\hat{P}_2 \tag{41}$$

Proof
For subsystem (34), let us consider the Lyapunov function candidate

$$V_2 = \hat{\xi}_2^{\mathrm{T}}\hat{P}_2(t)\hat{\xi} + \hat{\gamma}\tilde{\xi}^{\mathrm{T}}\tilde{P}_2\tilde{\xi} \tag{42}$$

Its time derivative along (35) and (36) is

$$\dot{V}_2 = -\hat{\xi}^{\mathrm{T}}Q_2\hat{\xi} - \hat{\xi}^{\mathrm{T}}\hat{P}_2B_2R_2^{-1}B_2^{\mathrm{T}}\hat{P}_2\hat{\xi} + 2\hat{\xi}^{\mathrm{T}}\hat{P}_2HC_2\tilde{\xi} - \hat{\gamma}\tilde{\xi}^{\mathrm{T}}C_2^{\mathrm{T}}C_2\tilde{\xi} \tag{43}$$

By completing the square,

$$2\hat{\xi}^{\mathrm{T}}\hat{P}_2HC_2\tilde{\xi} - \hat{\gamma}\tilde{\xi}^{\mathrm{T}}C_2^{\mathrm{T}}C_2\tilde{\xi} \leqslant \frac{1}{\hat{\gamma}^2}\hat{\xi}^{\mathrm{T}}\hat{P}_2HH^{\mathrm{T}}\hat{P}_2\hat{\xi}$$

Equation (43) becomes

$$\dot{V}_2 \leqslant -\hat{\xi}^{\mathrm{T}}Q_2\hat{\xi} - \hat{\xi}^{\mathrm{T}}\hat{P}_2B_2R_2^{-1}B_2^{\mathrm{T}}\hat{P}_2\hat{\xi} + \frac{1}{\hat{\gamma}^2}\hat{\xi}^{\mathrm{T}}\hat{P}_2HH^{\mathrm{T}}\hat{P}_2\hat{\xi} < 0 \tag{44}$$

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from which the exponential stability of the closed-loop system (34) can be concluded. In turn, we have the exponential convergence of state z.

To show optimality, substituting control u_2 in (38) with an incremental term Δu_2 (that is, $-R_2^{-1}B_2^{\mathrm{T}}\hat{P}_2\hat{\xi}+\Delta u_2$) into (40), we have

$$\begin{aligned}\hat{J}_2 &= \int_{t_0}^{\infty}\left\{[\hat{\xi}^{\mathrm{T}}\ \ \tilde{\xi}^{\mathrm{T}}]\hat{Q}_2(t)\begin{bmatrix}\hat{\xi}\\ \tilde{\xi}\end{bmatrix}\right\}\mathrm{d}t\\ &\quad+\int_{t_0}^{\infty}(\hat{\xi}^{\mathrm{T}}\hat{P}_2B_2R_2^{-1}B_2^{\mathrm{T}}\hat{P}_2\hat{\xi}-2\hat{\xi}^{\mathrm{T}}\hat{P}_2B_2R_2^{-1}\Delta u_2+\Delta u_2^{\mathrm{T}}R_2\Delta u_2)\,\mathrm{d}t\\ &= -\int_{t_0}^{\infty}\mathrm{d}V_2+\int_{t_0}^{\infty}\Delta u_2^{\mathrm{T}}R_2\Delta u_2\,\mathrm{d}t\\ &= V_2(\hat{\xi}(t_0),\tilde{\xi}(t_0))+\int_{t_0}^{\infty}\Delta u_2^{\mathrm{T}}R_2\Delta u_2\,\mathrm{d}t\end{aligned}\tag{45}$$

which is minimized by $\Delta u_2=0$. To this end, we have that the overall system is optimal with respect to $\hat{J}$. □

5.2. *Time-scaling method*

For the time-scaling method, we can also design the output-feedback counterpart of dynamic control (24). For the ease of observer-based control design, let us redesign dynamic-feedback control u_1 to be

$$\dot{u}_1=-\lambda(t)u_1-[\omega^2-0.25\lambda^2(t)]x_1\tag{46}$$

where $u_1(t_0)=c_u\|x(t_0)\|$. Similarly, substitute (46) into (2), it is easy to obtain the closed-loop solution as

$$\begin{aligned}x_1(t) &= \frac{1}{\sqrt{t-t_0+1}}\left\{x_1(t_0)\cos(\omega t-\omega t_0)+\frac{u_1(t_0)+0.5x_1(t_0)}{\omega}\sin(\omega t-\omega t_0)\right\}\\ u_1(t) &= -\frac{1}{2(t-t_0+1)^{3/2}}\left\{x_1(t_0)\cos(\omega t-\omega t_0)+\frac{u_1(t_0)+0.5x_1(t_0)}{\omega}\sin(\omega t-\omega t_0)\right\}\\ &\quad+\frac{\omega}{\sqrt{t-t_0+1}}\left\{-x_1(t_0)\sin(\omega t-\omega t_0)+\frac{u_1(t_0)+0.5x_1(t_0)}{\omega}\cos(\omega t-\omega t_0)\right\}\end{aligned}$$

To this end, let us introduce the following time transformation:

$$\begin{aligned}\tau &\triangleq \eta(t)=2\sqrt{t-t_0+1}-2\\ t &\triangleq \eta^{-1}(\tau)=0.25(\tau+2)^2+t_0-1\end{aligned}\tag{47}$$

It then follows from (3) and (47) that

$$\frac{\mathrm{d}z'}{\mathrm{d}\tau}=u_1'(\tau)A_2^*z'+B_2u_2'\tag{48}$$

where $z'(\tau)=z(t)$, $u_2'(\tau)=\sqrt{t-t_0+1}\,u_2(t)$, and $u_1'(\tau)=\sqrt{t-t_0+1}\,u_1(t)$.

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Since the pair $\{u_1'(\tau)A_2^*, B_2\}$ is uniformly completely controllable and the pair $\{u_1'(\tau)A_2^*, C_2\}$ is uniformly completely observable, we can then design the following time-varying observer:

$$\dot{\hat{z}}' = u_1'(\tau)A_2^*\hat{z}' + B_2u_2' + L(\tau)(z_1' - \hat{z}_1') \tag{49}$$

where $L(\cdot)$ is a time-varying gain vector to be selected. It follows from (48) and (49) that dynamics of estimation error $\tilde{z}' \triangleq z' - \hat{z}'$ are given by

$$\dot{\tilde{z}}' = [u_1'(\tau, x_0)A_2^* - L(\tau)C_2]\tilde{z}' \tag{50}$$

The following lemma provides a closed-form design of observer (49).

Lemma 5.2

Estimation error $\tilde{z}'$ of Equation (50) is globally and exponentially stable if gain vector $L(\tau)$ is set to be

$$L(\tau) = \begin{bmatrix} \delta_0 & \int_{\tau-\delta_0}^{\tau} \beta_\mu(s,\tau)\,\mathrm{d}s & \cdots & \int_{\tau-\delta_0}^{\tau} \dfrac{\beta_\mu^{n-2}(s,\tau)}{(n-2)!}\,\mathrm{d}s \\ \int_{\tau-\delta_0}^{\tau} \beta_\mu(s,\tau)\,\mathrm{d}s & \int_{\tau-\delta_0}^{\tau} \beta_\mu^2(s,\tau)\,\mathrm{d}s & \cdots & \int_{\tau-\delta_0}^{\tau} \dfrac{\beta_\mu^{n-1}(s,\tau)}{(n-2)!}\,\mathrm{d}s \\ \vdots & \vdots & \cdots & \vdots \\ \int_{\tau-\delta_0}^{\tau} \dfrac{\beta_\mu^{n-2}(s,\tau)}{(n-2)!}\,\mathrm{d}s & \int_{\tau-\delta_0}^{\tau} \dfrac{\beta_\mu^{n-1}(s,\tau)}{(n-2)!}\,\mathrm{d}s & \cdots & \int_{\tau-\delta_0}^{\tau} \dfrac{\beta_\mu^{2n-4}(s,\tau)}{(n-2)!(n-2)!}\,\mathrm{d}s \end{bmatrix}^{-1} C_2^{\mathrm{T}} \tag{51}$$

where $\beta_\mu(\tau, \tau_0) \triangleq \int_{\tau_0}^{\tau} u_1'(s, x_0)\,\mathrm{d}s$, $\delta_0 \geqslant \delta_{\mu,0}^*$ for some constant $\delta_{\mu,0}^*$.

Proof

By noting the uniformly complete observability of the pair $\{u_1'(\tau)A_2^*, C_2\}$, the proof is identical to that in Lemma 5 of [12]. □

It follows from (47) and (49) that the observer in the time t domain is

$$\frac{\mathrm{d}\hat{z}}{\mathrm{d}t} = \frac{\mathrm{d}\hat{z}'}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = u_1(t)A_2^*\hat{z} + B_2u_2 + \frac{1}{\sqrt{t-t_0+1}}L(\eta(t))(z_1 - \hat{z}_1) \tag{52}$$

Upon having the exponential stable observer (52), the dynamic control for u_2 be

$$u_2(t) = -\frac{1}{\sqrt{t-t_0+1}} r_2^{-1}(t)B_2^{\mathrm{T}}\hat{P}_2(t)\hat{z}(t) \tag{53}$$

where $\hat{P}_2(t) \triangleq \hat{P}_2'(\eta(t))$ is the solution to the following Riccati equation:

$$0 = \frac{\mathrm{d}\hat{P}_2'(\tau)}{\mathrm{d}\tau} + u_1'(\tau)\hat{P}_2'(\tau)A_2^* + (A_2^*)^{\mathrm{T}}\hat{P}_2'(\tau)u_1'(\tau) - \frac{1}{r_2'(\tau)}\hat{P}_2'(\tau)B_2B_2^{\mathrm{T}}\hat{P}_2'(\tau) + \hat{Q}_2 \tag{54}$$

Theorem 5.2

Under dynamic-feedback control (46) and (53) with observer (52), system (1) is globally asymptotically stable.

Proof
Consider the Lyapunov function candidate

$$V'(x_1,u_1,z,t) \triangleq V_1'(x_1,u_1,t)+V_2'(z,t)$$

where

$$V_1'(x_1,u_1,t) \triangleq \tfrac{1}{2}[u_1+0.5\lambda(t)x_1]^2+\tfrac{1}{2}\omega^2 x_1^2$$

and

$$V_2'(z,t) \triangleq z^{\mathrm{T}}\hat{P}_2(t)z$$

It follows from (46) and (2) that

$$\frac{\mathrm{d}V_1'}{\mathrm{d}t}=-0.5\lambda(t)[u_1+0.5\lambda(t)x_1]^2-0.5\lambda(t)\omega^2x_1^2=-\lambda(t)V_1'$$

On the other hand, note that

$$u_2(t)=-\frac{1}{\sqrt{t-t_0+1}}r_2^{-1}(t)B_2^{\mathrm{T}}\hat{P}_2(t)z(t)+\frac{1}{\sqrt{t-t_0+1}}r_2^{-1}(t)B_2^{\mathrm{T}}\hat{P}_2(t)\tilde{z}(t)$$

and

$$u_2'(\tau)=-\frac{1}{r_2'(\tau)}B_2^{\mathrm{T}}\hat{P}_2'(\tau)z'(\tau)+\frac{1}{r_2'(\tau)}B_2^{\mathrm{T}}\hat{P}_2'(\tau)\tilde{z}'(\tau)$$

It then follows that

$$\begin{aligned}\frac{\mathrm{d}V_2'(z,t)}{\mathrm{d}t}&=\frac{\mathrm{d}V_2'(z,t)}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t}=\frac{1}{\sqrt{t-t_0+1}}\frac{\mathrm{d}V_2''(z',\tau)}{\mathrm{d}\tau}\\
&=-\frac{1}{\sqrt{t-t_0+1}}(z')^{\mathrm{T}}\left[\hat{Q}_2+\frac{1}{r_2'}\hat{P}_2'B_2B_2^{\mathrm{T}}\hat{P}_2'\right]z'+\frac{2}{\sqrt{t-t_0+1}}(z')^{\mathrm{T}}\hat{P}_2'B_2\frac{1}{r_2'}B_2^{\mathrm{T}}\hat{P}_2'\tilde{z}'\\
&=-\frac{1}{\sqrt{t-t_0+1}}(z')^{\mathrm{T}}\left[\hat{Q}_2+\frac{1}{r_2'}\hat{P}_2'B_2B_2^{\mathrm{T}}\hat{P}_2'\right]z'+\frac{2}{\sqrt{t-t_0+1}}(z')^{\mathrm{T}}\hat{P}_2'B_2\frac{1}{r_2'}B_2^{\mathrm{T}}\hat{P}_2'\tilde{z}'\end{aligned}$$

To this end, recall the exponential stability of $\tilde{z}'(\tau)$, the exponential stability of $z(t)$ can easily be concluded by Lemma 2 of [15]. □

Following the similar discussions as those in Theorem 4.2 and procedures as those in Theorem 5.1 for state-scaling method, it is easy to show that controls (46) and (53) are optimal with respect to the performance indices $\hat{J}=\hat{J}_1+\hat{J}_2$, where

$$\hat{J}_1(t)=\int_{t_0}^{\infty}[x_1\ \ u_1]W_1(t)\begin{bmatrix}x_1\\u_1\end{bmatrix}\mathrm{d}t \tag{55}$$

and

$$\hat{J}_2'(\tau)=\int_{t_0}^{\infty}\left\{\begin{bmatrix}\hat{z}'(\tau)\\\tilde{z}'(\tau)\end{bmatrix}^{\mathrm{T}}W_2(\tau)\begin{bmatrix}\hat{z}'(\tau)\\\tilde{z}'(\tau)\end{bmatrix}+r_2'[u_2']^2\right\}\mathrm{d}\tau \tag{56}$$

with

$$W_1(t) = \begin{bmatrix} 0.25\lambda^3(t) + \lambda(t)\omega^2 & 0.5\lambda^2(t) \\ 0.5\lambda^2(t) & \lambda(t) \end{bmatrix}$$

$$W_2(\tau) = \begin{bmatrix} Q_2'(\tau) & -\hat{P}_2'(\tau)L'(\tau)C_2 \\ -C_2^{\mathrm{T}}L'(\tau)\hat{P}_2'(\tau) & \hat{\gamma}C_2^{\mathrm{T}}C_2 + \hat{\gamma}\Phi_0^{\mathrm{T}}C_2^{\mathrm{T}}C_2\Phi_0 \end{bmatrix}$$

and $\Phi_0(\tau-\delta,\tau)$ is the state transition matrix with respect to matrix $u_1'(\tau, x_0)A_2^*$, and positive constant $\hat{\gamma}'$ is chosen to satisfy

$$(\hat{\gamma}')^2 Q_2' > \hat{P}_2' L' L'^{\mathrm{T}} \hat{P}_2' \tag{57}$$

6. SIMULATION

In this section, simulation results are provided to illustrate the effectiveness of the proposed state-feedback and output-feedback controls, respectively.

6.1. State-feedback control results

The proposed continuous state-feedback controls are applied to stabilize a front-steering back-driving mobile robotic vehicle. Its kinematic model consists of four equations:

$$\dot{x}_c = \rho_c \cos(\theta_c)\omega_{c1}$$

$$\dot{y}_c = \rho_c \sin(\theta_c)\omega_{c1}$$

$$\dot{\theta}_c = \frac{\rho_c}{l_c}\tan(\phi_c)\omega_{c1}$$

$$\dot{\phi}_c = \omega_{c2}$$

where (x_c, y_c) are Cartesian coordinates of the guidepoint at the center of the rear wheel of the vehicle, θ_c is the orientation angle of the car body with respect to the x_c axis, ϕ_c is the steering angle, ρ_c is the driving wheel radius, l_c is the distance between the two wheel-axle centers, ω_{c1} is the angular velocity of the driving wheel, and ω_{c2} is the steering rate. Under the following transformations of coordinates and inputs [16, 17]:

$$x_1 = x_c$$

$$x_2 = y_c$$

$$x_3 = \tan(\theta_c)$$

$$x_4 = \frac{\tan(\phi_c)}{l\cos^3(\theta_c)}$$

and

$$\omega_{c1} = \frac{u_1}{\rho_c \cos(\theta_c)}$$

$$\omega_{c2} = -\frac{3\sin(\theta_c)}{l\cos^2(\theta_c)}\sin^2(\phi_c)u_1 + l\cos^3(\theta_c)\cos^2(\phi_c)u_2$$

the kinematic model can be transformed into chained form (1) with $n=4$.

In the simulation, initial condition of the state is set to be $x(t_0)=[0,-1,\pi,0]^{\mathrm{T}}$. In the proposed state-scaling control, design choices are set to be $r_1=1$, $q_1=10$, $r_2=1$, $q_2=10$, $\zeta=1$, $u_1(t_0)=10$, and $P_2(t)$ is set to be the steady-state solution to Riccati equation (A4). In the proposed time-scaling control, design choices are selected to be $r_2=1$, $q_2=10$, $\omega=1$, $u_1(t_0)=10$, and $P_2'(t)$ is set to be the steady-state solution to Riccati equation (25).

In Figure 1, simulation results of the proposed continuous controls under perfect feedback are shown. Clearly, despite of $x_1(t_0)=0$, exponential and asymptotic stabilities are achieved, respectively. The transient under the state-scaling control is somewhat larger but converges faster than that under the time-scaling control, which is a trade-off between transient overshoot and convergence rate. In Figure 2, simulation results are presented for the case that a small variation of $0.01*\sin(t)$ added to the measurements of x_2, x_3, and x_4 as noises, and they verify that the time-scaling control is computationally robust in the presence of measurement noises.

6.2. Output-feedback control results

For illustrating the output-feedback controls, let us consider the stabilization of third-order chained system. In the simulation, initial condition of the state is set to be $x(t_0)=[0,-0.5,0.5]^{\mathrm{T}}$. In the proposed state-scaling output-feedback control, design choices are set to be $r_1=1$, $q_1=10$, $r_2=1$, $q_2=10$, $\zeta=1$, $u_1(t_0)=1$. The initial conditions for observer is set to $[0,0]^{\mathrm{T}}$. In the proposed time-scaling control, design choices are selected to be $r_2=0.1$, $q_2=50$, $\omega=0.2$, $u_1(t_0)=0.1$, and

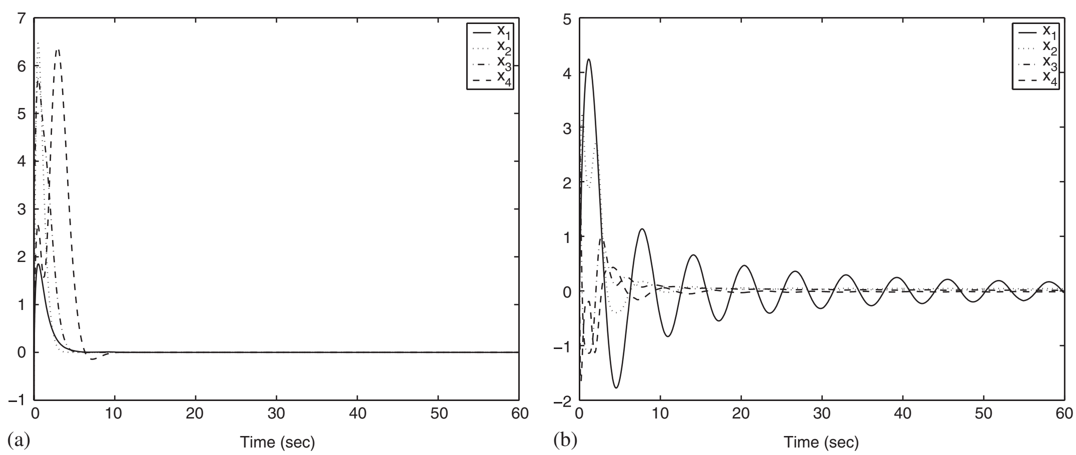

Figure 1. Simulation in the absence of any measurement noise: (a) state trajectories under the state-scaling control and (b) state trajectories under the time-scaling control.

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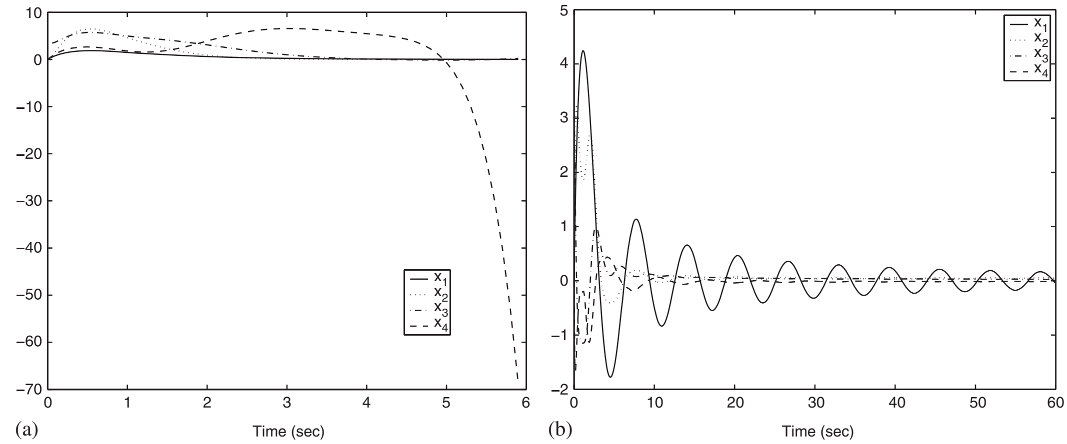

Figure 2. Simulation results in the presence of a small measurement noise: (a) state responses under the state-scaling control and (b) state responses under the time-scaling control.

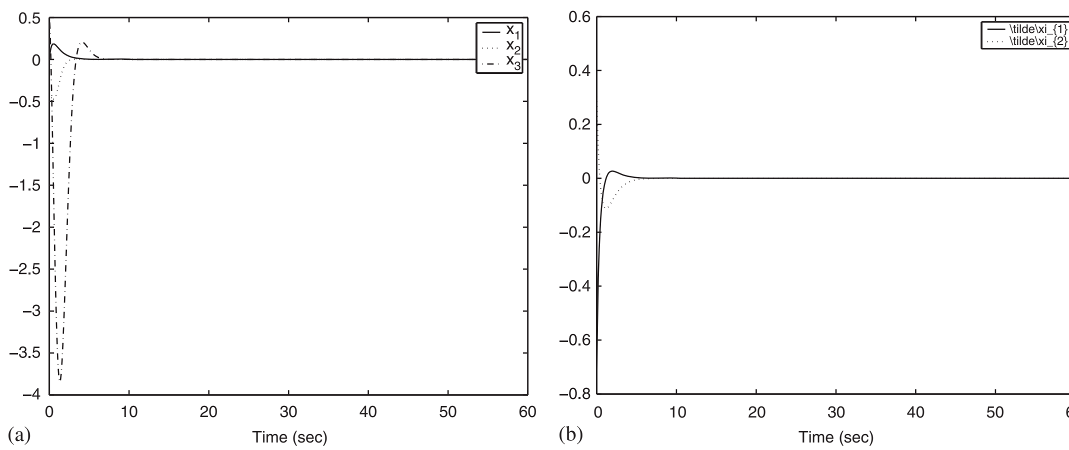

Figure 3. Simulation results for state-scaling output-feedback control: (a) state responses under the output-feedback state-scaling control and (b) state estimation errors.

$P_2(t)$ is set to be the steady-state solution to Riccati equation (25). The choices for observer are: initial condition $\hat{z}(t_0)=[0,0]^{\mathrm{T}}$, $\delta_0=2$, and observer gain vector $L(t)$ in (51). Simulation results in Figures 3–6 illustrate the effectiveness of the proposed designs.

7. CONCLUSION

In this paper, feedback stabilization problem of nonholonomic chained systems is studied by investigating uniform complete controllability and developing relevant results. It is shown using

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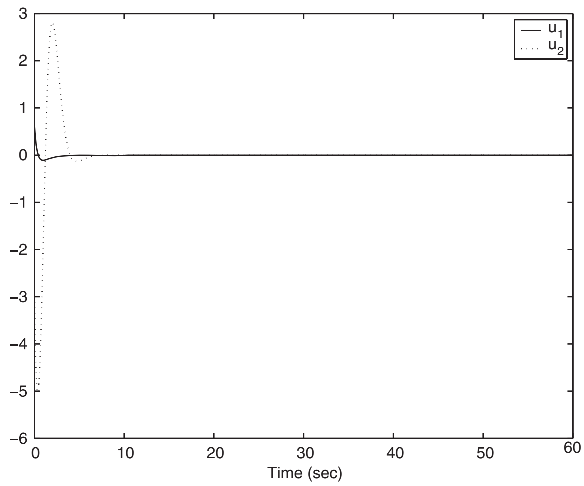

Figure 4. State-scaling output-feedback controls.

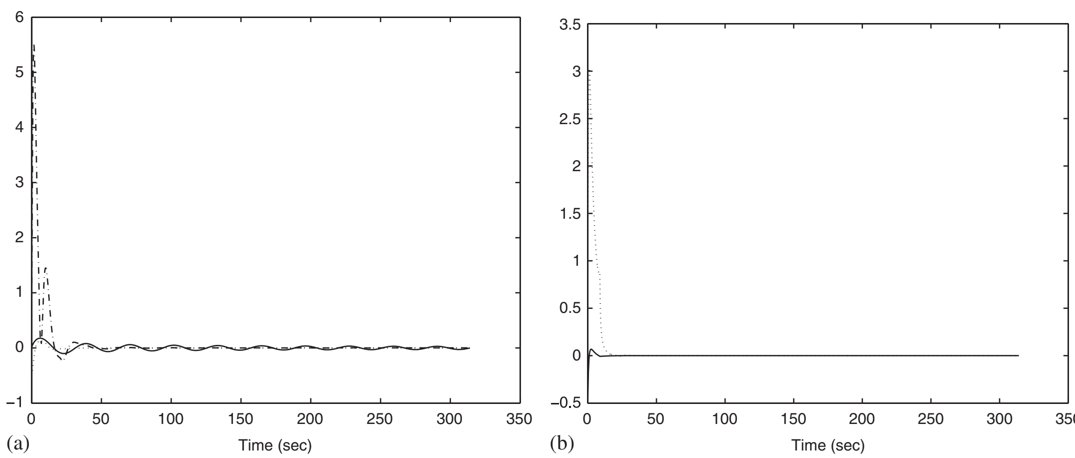

Figure 5. Simulation results for time-scaling output-feedback control: (a) state responses under the output-feedback time-scaling control and (b) state estimation errors.

two illustrative examples that linear controllability does not hold for stabilization of the chained system but can be recovered under either a state-scaling transformation or a time-scaling transformation. Based on the idea of recovering linear controllability, two new design methodologies using state- and time-scaling transformations are proposed, and both of them require innovative designs of dynamic control component u_1. The newly proposed state-scaling transformation is globally singularity free and enables the design of a continuous exponentially convergent control, and the time-scaling method yields a continuous asymptotically stabilizing control without the need of using any state transformation. Both methods are used in the state-feedback and output-feedback

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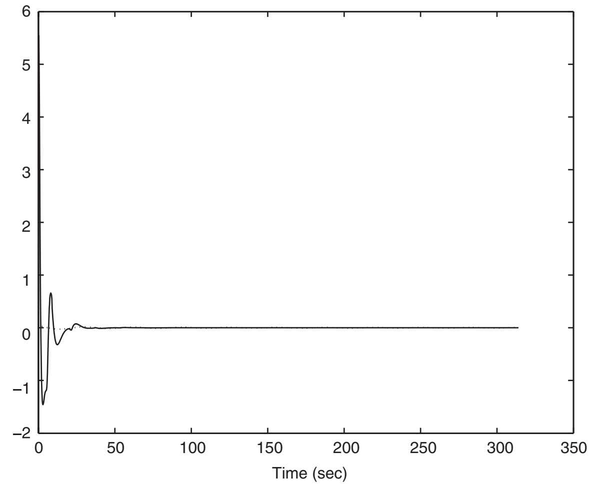

Figure 6. Time-scaling output-feedback controls.

control designs. All the results are shown to be systematic and straightforward as well as to render controls of optimal performance.

APPENDIX A: UNIFORM COMPLETE CONTROLLABILITY

Chained system (1) can be expressed as the following LTV system:

$$\dot{x}_1 = u_1, \quad \dot{z} = u_1(t) A_2^* z + B_2 u_2 \tag{A1}$$

where matrices A_2^* and B_2 are those defined in (4), $x = [x_1 \;\; z^{\mathrm{T}}]^{\mathrm{T}} \in \Re^n$ is the state, and u_1 and u_2 are control input components. Although control u_1 can easily be designed, it is shown in [12] that systematical design of control u_2 calls for thorough study of controllability. Specifically, although system (A1) is known to be nonlinear small-time controllable [3], better controllability property needs to be developed for systematic control design. To this end, consider the following generic LTV system:

$$\dot{\xi} = F_2(t)\xi + B_2 u_2 \tag{A2}$$

where $F_2(t)$ is uniformly bounded. Let us define

$$\begin{aligned} \dot{\Phi}(t, t_0) &= F_2(t)\Phi(t, t_0) \\ W_c(t_0, t_f) &= \int_{t_0}^{t_f} \Phi(t_0, \tau) B_2 B_2^{\mathrm{T}} \Phi^{\mathrm{T}}(t_0, \tau)\, \mathrm{d}\tau \end{aligned} \tag{A3}$$

which can be viewed as open-loop state transition matrix and controllability Grammian of system (A2), respectively. Hence, the following standard definition can now be adopted from [12, 18].

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Definition A.1 (*Kalman* [*18*])
System (A2) is uniformly completely controllable if the inequalities

$$0<\alpha_{c1}(\delta)I\leqslant W_c(t,t+\delta)\leqslant\alpha_{c2}(\delta)I$$

and

$$\|\Phi(t,t+\delta)\|\leqslant\alpha_{c3}(\delta)$$

hold for some constant $\delta>0$, for some fixed positively valued functions $\alpha_{ci}(\cdot)$, and for all t.

Definition A.2 (*Qu et al.* [*12*])
A time function $w(t):[t_0,\infty)\rightarrow\Re$ is said to be uniformly right continuous if, for every $\varepsilon>0$, there exists $\eta>0$ such that $t\leqslant s\leqslant t+\eta$ implies $|w(s)-w(t)|<\varepsilon$ for all $t\in[t_0,\infty)$. Time function $w(t):[t_0,\infty)\rightarrow\Re$ is said to be uniformly nonvanishing if there exist constants $\delta>0$ and $\underline{w}>0$ such that, for any value of t, $|w(s)|\geqslant\underline{w}$ holds somewhere within the interval $[t,t+\delta]$.

The following lemmas summarize the relevant results in [12].

Lemma A.1 (*Kalman* [*18*] *and Qu et al.* [*12*])
Consider the solution to the following differential Riccati equation: for some $P_2(\infty)>0$ and for any given $0<\underline{q}_2\leqslant q_2(t)\leqslant\overline{q}_2$ and $0<\underline{r}_2\leqslant r_2(t)\leqslant\overline{r}_2$

$$0=\dot{P}_2(t)+P_2(t)F_2(t)+F_2^{\mathrm{T}}(t)P_2(t)+C_2^{\mathrm{T}}q_2(t)C_2-P_2(t)B_2r_2^{-1}(t)B_2^{\mathrm{T}}P_2(t) \tag{A4}$$

If both pairs $\{F_2(t),B_2\}$ and $\{F_2^{\mathrm{T}}(t),C_2^{\mathrm{T}}\}$ are uniformly completely controllable, solution $P_2(t)$ exists and is uniformly bounded, $V(\xi,t)=\xi^{\mathrm{T}}P(t)\xi$ is positive definite, and $\xi^{\mathrm{T}}[C_2^{\mathrm{T}}q_2(t)C_2+P_2(t)B_2r_2^{-1}(t)B_2^{\mathrm{T}}P_2(t)]\xi$ is also positive definite.

Lemma A.2 (*Qu et al.* [*12*])
Suppose that component $u_1(t)$ is designed to be uniformly right continuous, uniformly bounded, and uniformly nonvanishing. Then, system (A1) is uniformly completely controllable.

To solve the problem of trajectory tracking, $u_1(t)$ being nonvanishing can usually be assumed, and state-feedback and output-feedback controls are designed in [12] to ensure near optimality. For stabilization (and regulation), control component $u_1(t)$ has to be vanishing. For the case that $u_1(t)$ is vanishing, it can be shown that system (A1) is not uniformly completely controllable, detailed analysis and control design are *not* carried out in [12] except for providing the following two interesting examples. Particularly, Example A.1 shows that controllability of (A1) can be recovered by state transformation (state scaling), whereas Example A.2 provides a case of using time folding (time scaling) to recover the controllability of (A1).

Example A.1
Consider system (A1) with

$$u_1(t)=\frac{1}{\kappa(t)}u_1'(t) \tag{A5}$$

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where $u_1'(t)$ is uniformly nonvanishing, $\kappa(t)>0$ for any finite time $t \geqslant t_0$, $\lim_{t\to\infty}\kappa(t)=+\infty$, and $1/\kappa(t)\in L_1$. Obviously, signal $u_1(t)$ is vanishing. Nonetheless, uniform complete controllability can be recovered by a state transformation. For instance, consider system (A1) with (A5),

$$\kappa(t)=e^t, \quad u_1'(t)=1, \quad A_2^* = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Under the time-unfolding state transformation:

$$z = \begin{bmatrix} e^{-t} & 0 \\ 0 & 1 \end{bmatrix} z'$$

subsystem of z in (A1) is transformed into

$$\dot{z}' = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} z' + B_2 u_2$$

which is uniformly completely controllable and can be exponentially stabilized.

Example A.2

Consider system (A1) with $u_1(t)$ defined by (A5) except that $1/\kappa(t)\notin L_1$. Although signal $u_1(t)$ is vanishing, uniform complete controllability can be recovered by a simple time-scaling transformation. Specifically, consider the case that $\kappa(t)=\sqrt{t-t_0+1}$. Let us introduce the following time scaling:

$$\tau = 2\sqrt{t-t_0+1}-2$$

Under the time-scaling transformation, the subsystem of z in (A1) is mapped into

$$\frac{\mathrm{d}z(\tau)}{\mathrm{d}\tau} = u_1' A_2^* z(\tau) + B_2 u_2' \tag{A6}$$

where $u_2' = \sqrt{t-t_0+1}\cdot u_2$. Clearly, system (A6) is uniformly completely controllable for exponential stabilization.

The objective of this paper is to show that the ideas exposed by the above examples render new and systematic designs of continuous and optimal controls for stabilizing nonholonomic chained systems.

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