Nonlinear Simulation of a String System Under Boundary Robust Control

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Abstract

In this paper, a new boundary robust vibration control is designed for a nonlinear string system using the Lyapunov direct method. Simulations are carried out using the finite difference approach, and open-loop and closed-loop responses of the nonlinear system are compared to demonstrate the effectiveness and robustness of the proposed boundary control. Uncertainties considered in the design and simulation include such dynamics as nonlinear string tension, string non-uniformity, and nonlinear friction. The simulation results demonstrate that, in the presence of significant uncertainties, any vibration of the string can be exponentially suppressed by the proposed robust control.

Nomenclature

\( \{x_0, y_0, z_0\}, \{x, y, z\}, t \) — inertia frame, coordinate system fixed onto the transporter, and time

\( x, dx \) — axial coordinate along the equilibrium of the string, and an element along the axis

\( y(x, t) \) — transverse displacement with respect to the equilibrium of the string (w.r.t. the transporter)

\( y_1, y_2, \dot{y}_1, \dot{y}_2, y_{xx} = \frac{\partial y(x, t)}{\partial t}, \frac{\partial y(x, t)}{\partial x}, \frac{\partial^2 y(x, t)}{\partial x^2}, \frac{\partial^2 y(x, t)}{\partial t \partial x}, \frac{\partial^2 y(x, t)}{\partial x^2} \)

\( A(z), \rho(z), m(z) = \rho(z)A(z) \) — cross-section area, linear density of the string, and the mass per unit length

\( E, l \) — elastic modulus, and axial length between supports

\( T_0(z), T(z, t) \) — string initial tension, and nonlinear tension in the string

\( y_0(t), \dot{y}_0(t), \ddot{y}_0(t) \) — position, velocity, and acceleration of the moving transporter

\( M_0, M_1, p_0(t), p_1(t) \) — masses and positions of the control mechanism (at \( z = 0, l \) w.r.t. \( \{x, y, z\} \))

\( b_0(p_0, t), b_1(p_1, t) \) — nonlinear dynamic friction “coefficients” between the control mechanism and the transporter

\( f_0(t), f_1(t) \) — boundary control forces

\( \Delta x, \Delta t \) — sample increments in \( x \) and \( t \), respectively

\( w^*_n \) — value of a spatial function \( w(x, t) \) at the sample point \( (x, t) = (k\Delta x, n\Delta t) \)

1 Introduction

String-like systems and their vibration suppression have attracted a lot of research interests due to the fact that many physical plants such as telephone wires, cables, conveyor belts, and even human DNA can be modeled by strings. Although extensive research has been done in string modeling and vibration control (for example, [16, 3, 9, 10, 4, 14, 15, 12, 11] and the references therein), boundary robust adaptive control of nonlinear string systems was first proposed in [8]. Later, robust adaptive control of string system was simulated in [5] using the modal decomposition technique (which inherently assumes a linear or quasi-linear model for string dynamics).

This paper presents results on vibration control for nonlinear, uncertain string models. Specifically, an explicit robust control is designed for an uncertain string whose nonlinear dynamics vary within a range centered at their nominal model. It is an extension of the one in [8] that robust control is designed in the presence of uncertain dynamics of control mechanism. Compared to [5], simulation study and performance evaluation are done using the finite difference method so that nonlinear uncertainties in the partial differential equation are taken into account.

The robust control is designed for control forces applied to the ends of string through actuators, and it needs three boundary feedbacks of velocity, slope, and slope rate. It is shown analytically that string vibration is damped out exponentially under the proposed control. To demonstrate effectiveness of the robust boundary control, the finite difference approach is introduced to approximate partial derivatives at
various isolated points and to carry out dynamic simulations. As none of commercially available packages is capable of simulating nonlinear partial differential equations with boundary controls, a set of Matlab macros have been written. In the presentation of simulation results, displacements and velocity at several points along the string are plotted. To quantify distributed responses of the closed-loop system, an $L_2$ spatial norm is calculated to show that the whole string exponentially converges to equilibrium.

2 System Model

![Figure 1: A stretched string on a moving transporter](image)

In this paper, vibration control of a nonlinear flexible system, shown in figure 1, is considered. The system, extracted from material handling in manufacturing, consists of a nonlinear stretched string, two supporting/rotation mechanisms, and a transporter. It follows from the Newton’s Law and from the string model in (2) that dynamic equations governing the motion of the system are:

$$M_0[\ddot{p}_0(t) + \ddot{y}_b] = f_0(t) - T(0,t)y_z(0,t) - b_0(p_0,t)p_0(t), \quad (1)$$

$$M_1[\ddot{p}_l(t) + \ddot{y}_b] = f_l(t) + T(l,t)y_z(l,t) - b_l(p_l,t)p_l(t), \quad (2)$$

and

$$m(x)\frac{\partial^2 Y(x,t)}{\partial x^2} = \frac{\partial}{\partial x}\left\{T(x,t)\frac{\partial Y(x,t)}{\partial x}\right\}. \quad (3)$$

where $Y(x,t) = y(x,t) + y_b(t)$ represents displacement of the string with respect to ground. It is assumed in this paper that the string tension is nonlinear and of form

$$T(x,t) = T_0(x) + w(x)y_z^2(x,t), \quad (4)$$

where $T_0(x) > 0$ and $w(x) \geq 0$ for all $x \in [0, l]$. In terms of $y(x,t)$, string model (3) can be rewritten as

$$m(x)y_{tt} = \frac{\partial T_0(x)}{\partial x}y_z(x,t) + \frac{\partial w(x)}{\partial x}y_z^2(x,t) + [T_0(x) + w(x)y_z^2(x,t)]y_z(x,t) - m(x)y_t(x,t). \quad (5)$$

The system under consideration and in figure 1 is described by nonlinear ordinary differential equations (1) and (2) together with nonlinear partial differential equation (5). In the model, forces $f_0(t)$ and $f_l(t)$ are boundary control variables to be designed. Given $f_0(t)$ and $f_l(t)$, trajectory of the system can be solved under the initial conditions (initial positions of control mechanisms, and initial displacement and velocity of the string)

$$p_0(0), \quad p_l(0), \quad y(x,0) \quad y_z(x,0), \quad (6)$$

and under the boundary conditions

$$y(0,t) = p_0(t), \quad y(l,t) = p_l(t). \quad (7)$$

3 Vibration Control Problem

In this paper, the problem of using boundary controls to suppress vibration is considered. The difference between the proposed problem and those studied in the past is that the system under consideration may contain significant uncertainties as defined by the following assumption.

**Assumption 1** Functions $T_0(x)$, $w(x)$, and $m(x)$ are uncertain functions of form

$$T_0(x) = T_n + \Delta T_0(x), \quad w(x) = w_0 + \Delta w(x),$$

$$m(x) = m_0 + \Delta m(x),$$

where $T_n > 0$, $w_0 \geq 0$ and $m_0 > 0$ are the nominal values. Functions $b_0(p_0,t)$, $b_l(p_l,t)$, $\Delta T_0(x)$, $\Delta w(x)$ and $\Delta m(x)$ represent the uncertainties in the system dynamics, and they and some of their first-order partial derivatives are bounded as: for some constants $0 \leq c_1 < 1$ and $c_2$, $c_3$, $c_4 \geq 0$,

$$|\Delta T_0(x)| \leq c_1 T_n, \quad |\Delta w(x)| \leq c_1 w_0, \quad |\Delta m(x)| \leq c_1 m_0,$$

$$0 \leq b_0(p_0,t) \leq c_2, \quad 0 \leq b_l(p_l,t) \leq c_2,$$

$$\left|\frac{\partial \Delta T_0(x)}{\partial x}\right| \leq c_3 T_n, \quad \left|\frac{\partial \Delta w(x)}{\partial x}\right| \leq c_3 w_0,$$

and

$$\left|\frac{\partial \Delta m(x)}{\partial x}\right| \leq c_3 m_0.$$
Boundary Robust Control Problem: Find controls $f_0(t)$ and $f_1(t)$ such that, in the presence of uncertainties as defined in assumption 1, any vibration of nonlinear string (3) can be suppressed asymptotically (i.e., $Y(x, \infty) = 0$ for all $x \in [0, l]$) using only boundary feedback information.

4 Robust Control Design

In what follows, a robust boundary controller is designed by Lyapunov direct method to solve the vibration control problem proposed in the last section. Such a control and its stability property are given by the following theorem.

**Theorem 1** Consider the nonlinear string system consisting of differential equations (1) and (2), partial differential equation (5), and boundary conditions in (7). Under assumption 1, and the following boundary controls are robust, and they ensure exponential (and global for all initial conditions defined in (6)) stability with respect to the equilibrium of the string:

$$f_0(t) = -k_0 Y_2(0, t) + 3T_n + w_0 y_0^2(0, t) + 3z(0, t) + c_2 y_1(0, t) |\text{sign}(y_1(0, t))|,$$

$$f_1(t) = -k_1 Y_1(l, t) - 3T_n + w_0 y_0^2(l, t) + c_2 y_1(l, t) |\text{sign}(y_1(l, t))|,$$

where $\text{sign}()$ is the standard sign function, and $k_0 > 0$ and $k_1 = \sqrt{(1 - c_1 T_n)} m_0$ are positive control gains.

Proof: Let Lyapunov functional candidate for the system be

$$V(t) = V_0(t) + V_1(t) + V_2(t),$$

where $V_0(t)$, $V_0(t)$ and $V_1(t)$ are defined to be

$$V_0(t) = \frac{1}{2} m_0 \left[ y_0(t) + y_0(t) \right]^2,$$

$$V_1(t) = \frac{1}{2} M_1 \left[ y_1(t) + y_1(t) + \frac{3}{8} \alpha(t) y_2(t) \right]^2,$$

and function $\alpha(x)$ is given by

$$\alpha(x) = \sqrt{\frac{(1 - c_1 T_n)}{1 + c_1 m_0}} e^{-c_1 T_n (x - l)}.$$

It is obvious that $V_0(t) \geq 0$ and $V_1(t) \geq 0$. It follows from inequality $a^2 + b^2 \geq 2ab$ that $V_2(t)$ positive definite (with respect to $Y_r(x, t)$ and $y_2(x, t) = Y_2(x, t)$) as

$$V_2(t) \leq \frac{1 + c_1}{2} \max \left\{ m_0, T_n \right\} \int_0^l \left\{ y_1^2(x, t) + y_2^2(x, t) \right\} dx,$$

and

$$V(t) \leq \frac{1 + c_1}{2} \max \left\{ m_0, T_n, w_0 \right\} \int_0^l \left\{ y_1^2(x, t) + y_2^2(x, t) \right\} dx.$$

It follows from differential equations (1) and (2), from boundary conditions in (7), and from boundary robust controls (8) and (9) that the time derivatives of $V_0(t)$ and $V_1(t)$ are

$$V_0 = \left\{ \begin{array} {l} Y_2(0, t) \left[ f_0(t) - T(0, t) y_2(0, t) \right] \\
- b_0 Y_2(0, t) \end{array} \right.$$

$$= \frac{-2k_0}{M_0} V_0 + 2Y_1(0, t) T(0, t) y_2(0, t) + \left( Y_1(0, t) \left[ f_0(t) + 2k_0 Y_1(0, t) \right] \\
- 3T(0, t) y_2(0, t) - b_0 Y_2(0, t) \right)$$

$$\leq \frac{-2k_0}{M_0} V_0 + 2Y_1(0, t) T(0, t) y_2(0, t),$$

and, similarly,

$$V_1 = \left\{ \begin{array} {l} Y_1(l, t) + \frac{3}{8} \alpha(t) y_2(l, t) \left[ T(l, t) y_2(l, t) \right] \\
- b_1 y_1(l, t) \end{array} \right.$$

$$- \left( Y_1(l, t) + \frac{3}{8} \alpha(t) y_2(l, t) \left[ T(l, t) y_2(l, t) \right] \\
- \frac{2k_1}{M_1} Y_1 \right).$$

On the other hand, it follows from partial differential equation (3) that the time derivative of $V_4(t)$
\[ 
\dot{V}_s(t) = \int_0^t \left( \frac{\alpha(x)m(x)x}{l} Y_t(x,t)y_{xx}(x,t) \\
+ 2m(x)Y_t(x,t)Y_{tt}(x,t) \\
+ 2T(x,t)y_x(x,t)y_{xx}(x,t) \\
+ \frac{\alpha(x)m(x)x}{l} Y_t(x,t)y_{xx}(x,t) \right) \, dx \\
= \int_0^t \left\{ 2Y_t(x,t) \left[ \frac{\partial T_0(x)}{\partial x} y_x(x,t) + \frac{\partial w(x)}{\partial x} y_x^2(x,t) \right] \\
+ \frac{\alpha(x)x}{l} Y_t(x,t)y_{xx}(x,t) \right\} \, dx \\
= \int_0^t \left\{ \frac{\partial}{\partial x} \left[ 2Y_t(x,t)y_x(x,t)Y_x(x,t) \right] \\
+ 1 \frac{\alpha(x)m(x)x}{l} \frac{\partial Y_x(x,t)}{\partial x} \\
+ \frac{\alpha(x)x}{l} \frac{\partial T_0(x)}{\partial x} y_x^2(x,t) \\
+ \frac{3}{4} \frac{\alpha(x)x}{l} \frac{\partial w(x)}{\partial x} y_x^4(x,t) \\
+ \frac{1}{4} \frac{\alpha(x)x}{l} \frac{\partial w(x)}{\partial x} y_x^4(x,t) \right\} \, dx. 
\]

Integrating by parts yields
\[ 
\dot{V}_s(t) = -2T(0,t)y_x(0,t)Y_t(0,t) \\
+ 2T(t,y_x(l,t))Y_t(l,t) + \frac{3}{8} \alpha(l)y_x(l,t) \\
- \int_0^l \frac{1}{2l} \left[ \frac{\partial (\alpha(x)x)}{\partial x} \frac{\partial T_0(x)}{\partial x} - \alpha(x)x \frac{\partial T_0(x)}{\partial x} \right] \\
y_x^2(x,t) \, dx - \frac{1}{4} \alpha(l)T_0(l)y_x^2(l,t) \\
- \int_0^l \frac{1}{2l} \frac{\alpha(x)m(x)x}{l} Y_t(x,t) \, dx \\
+ \frac{1}{2} \alpha(l)m(l)Y_t^2(l,t) - \int_0^l \frac{1}{4l} \left[ 3 \frac{\partial (\alpha(x)x)}{\partial x} w(x) - \alpha(x)x \frac{\partial w(x)}{\partial x} \right] y_x^2(x,t) \, dx. 
\]

It is straightforward to verify that, under assumption 1, the choice of \( \alpha(x) \) ensures the following inequality: for all \( x \in [0,l] \),
\[ 
\frac{\partial (\alpha(x)m(x)x)}{\partial x} \geq 2\epsilon, \\
\frac{\partial (\alpha(x)x)}{\partial x} T_0(x) - \alpha(x)x \frac{\partial T_0(x)}{\partial x} \geq 2\epsilon, \\
and \\
3 \frac{\partial (\alpha(x)x)}{\partial x} w(x) - \alpha(x)x \frac{\partial w(x)}{\partial x} \geq 4\epsilon, 
\]
where
\[ 
\epsilon = \frac{1}{2l} \sqrt{\frac{(1 - c_1)T_n}{(1 + c_1)m}} e^{-\frac{c_3}{2}T} \max \{ (1 - c_1)m, (1 - c_1)T_n, 1.5(1 - c_1)w_0 \} > 0. 
\]

Therefore, we have
\[ 
\dot{V}_s(t) \leq -2T(0,t)y_x(0,t)Y_t(0,t) \\
+ 2T(t,y_x(l,t))Y_t(l,t) + \frac{3}{8} \alpha(l)y_x(l,t) \\
- \frac{1}{4} \alpha(l)T_0(l)y_x^2(l,t) + \frac{1}{2} \alpha(l)m(l)Y_t^2(l,t) \\
- \epsilon \int_0^l \left[ Y_t^2(x,t) + y_x^2(x,t) \right] \, dx. 
\]

Combining inequalities (13), (11) and (12) yields
\[ 
\dot{V}(t) \leq -2k_0 V_0 - \frac{2k_1}{M_i} V_1 - \frac{1}{4} \alpha(l)T_0(l)y_x^2(l,t) \\
+ \frac{1}{2} \alpha(l)m(l)Y_t^2(l,t) - \epsilon \int_0^l \left[ Y_t^2(x,t) + y_x^2(x,t) \right] \, dx. 
\]

It again follows from inequality \( a^2 + b^2 \geq 2ab \) and from the definition of \( \alpha(x) \) that
\[ 
- \frac{k_1}{M_i} V_1 - \frac{1}{4} \alpha(l)T_0(l)y_x^2(l,t) \\
+ \frac{1}{2} \alpha(l)m(l)Y_t^2(l,t) \\
= -\frac{1}{2} \left[ \frac{k_1 - \alpha(l)m(l)}{k_1 + \alpha(l)m(l)} \right] Y_x^2(l,t) - \frac{9}{128} k_1 \alpha(l)^2 \\
+ \frac{1}{4} \alpha(l)T_0(l) y_x^2(l,t) - \frac{3}{8} k_1 \alpha(l) Y_t(l,t) y_x(l,t) \\
\leq -\sqrt{k_1 - \alpha(l)m(l)} \sqrt{\frac{9}{64} k_1 \alpha^2(l) + \frac{1}{2} \alpha(l) T_0(l) \times} \\
\left[ Y_t(l,t) y_x(l,t) - \frac{3}{8} k_1 \alpha(l) Y_t(l,t) y_x(l,t) \right] \\
\leq 0. 
\]
Substituting the above inequality and invoking (10), we have
\[
V(t) \leq \frac{-2k_0 M_0 V_0 - k_l M_l V_t - \epsilon}{M_0} \int_0^t \{ \dot{V}_t^2(x, t) + y_2^2(x, t) + y_3^2(x, t) \} \, dx
\]
from which exponential stability can be concluded.

5 Nonlinear Simulation

Nonlinear differential equations of the system and nonlinear boundary controls have been described in the previous section. None of the existing packages is capable of simulating such a controlled system. Thus, a set of Matlab macros were developed. Due to the nonlinear and distributed nature of the system, finite difference approach is employed in the simulation study.

5.1 Finite Difference Approach

The basic idea of the finite difference approach is to convert a continuous-time system into an equivalent discrete system suitable for computer calculation. Ordinary differential equation can be easily discretized in time into a discrete time system. To find an approximate solution of a partial differential equation, approximations of various partial derivatives at isolated points must also be developed over the same domain as the original equation. In the latter, one has to decide what approximations are to be used and their errors. In this paper, the following approximations are used: For any scalar multi-variable function \( w(\cdot, s, \cdot) \), let \( i \) be the discrete index along the s axis and let \( \Delta s \) the step-size, then

\[
w(\cdot, i\Delta s + \Delta s, \cdot) = w(\cdot, i\Delta s, \cdot) + \Delta s \frac{\Delta^2 w}{\Delta s^2},
\]
\[
w(\cdot, i\Delta s, \cdot) = w(\cdot, i\Delta s + \Delta s, \cdot) - w(\cdot, i\Delta s, \cdot) + \Delta s \frac{\Delta^2 w}{\Delta s^2}
\]
\[
w(\cdot, i\Delta s, \cdot) = w(\cdot, i\Delta s, \cdot) - w(\cdot, i\Delta s - \Delta s, \cdot) + \Delta s \frac{\Delta^2 w}{\Delta s^2},
\]
and
\[
w_{x_s}(\cdot, i\Delta s, \cdot) = \left[ w(\cdot, i\Delta s + \Delta s, \cdot) - 2w(\cdot, i\Delta s, \cdot) + w(\cdot, i\Delta s - \Delta s, \cdot) \right] / \Delta s^2 + \Delta (\Delta s^2)
\]
where \( a = O(b) \) stands for infinitesimals of the same order (i.e., \( |a| \leq c|b| \) for some constant \( c > 0 \) and for all \( b \)). Approximation (15) is of second order, and is referred to as a central difference formula. Equations (16) and (17) are called forward difference and backward difference approximations, and they are of first order. Equation (18) is the elementary approximation of a second order partial derivative. These approximations can be easily verified, as did in [1], using Taylor series expansion.

In an application of the finite difference approach, computational molecules need to be constructed. For the string system, let \( k \) and \( n \) be the indices along the x and t axes, respectively; let \( \Delta x \) and \( \Delta t \) be the stepsizes, respectively; let \( N_k \) be the number of molecules and \( N_t \) be the simulation time, and let \( y_{k} \) be the value of \( y(x, t) \) at \( (k\Delta x, n\Delta t) \). Then, three sets of equations need to be developed. The first set deals with both time dependence and spatial dependence but not boundary conditions or initial conditions. Applying approximation (15) to \( y_{k}(x, t) \), \( y_{k+2}(x, t) \) and \( y_{b}(t) \) and using approximation (18) for first-order partial derivatives with respect to \( s \), we can rewrite system equation (5) into following finite difference equation: for \( k = 1, 2, 3, \ldots, N_x - 1 \) and for \( n = 1, 2, \ldots, N_t \),

\[
y_{k+1} = \frac{\Delta t^2}{4m(k\Delta x)\Delta z^2} \left[ T_b[(k + 1)\Delta z] - T_b[(k - 1)\Delta z] \right] (y_{k+1} - y_{k-1})
\]
\[
+ \frac{\Delta t^2}{16\Delta x^2 m(k\Delta x)} [w[(k + 1)\Delta z] - w[(k - 1)\Delta z]] (y_{k+1}^n - y_{k-1}^n)^3
\]
\[
+ \frac{\Delta t^2}{m(k\Delta x)\Delta z^2} (y_{k+1}^n - 2y_{k}^n + y_{k-1}^n) \times
\]
\[
\left[ T_b(k\Delta x) + 3w(k\Delta x) \left( \frac{y_{k+1}^n - y_{k-1}^n}{2\Delta z} \right)^2 \right] - [y_{b}(n\Delta t + \Delta t) - y_{b}(n\Delta t)]
\]
\[
+ y_{b}(n\Delta t - \Delta t) + 2y_{k}^n - y_{k-1}^n,
\]
where \( \Delta x = l/(N_x - 1) \), and \( \Delta t \) is a given small positive constant stepsize such that \( N_t = T/\Delta t + 1 \) is an integer.

Equation (19) can be used to calculate \( y_{k+1}^n \) (with respect to spatial index \( k \)) from the values of its neighboring points except for the cases that \( k = 0 \) and \( k = N_x \) as \( y_{1}^n \) and \( y_{N_x}^n \) do not exist. The
values of \( y_k^{n+1} \) at \( k = 0 \) and \( k = N_x \) are defined by boundary conditions in (7) together with ordinary differential equations (1) and (2). At \( k = 0 \) or \( x = 0 \), we use central difference approximation (15) for \( \bar{y}_0(t) = y_{tt}(0,t), y_0(t), \) and \( \bar{y}_0(t) \) and apply forward approximation (16) to \( y_{x}(0,t) \). By doing so, we now rewrite differential equation (1) as

\[
y^{n+1}_0 \cong \left[ M_0 + \frac{\Delta t}{2} b_0(y^{n}_0, n\Delta t) \right]^{-1} \left[ 2M_0 y^{n}_0 - M_0 y^{n-1}_0 - M_0 y_{x}(n\Delta t + \Delta t) - 2y_0(n\Delta t - \Delta t) + \Delta t^2 f_0(n\Delta t) - \frac{\Delta t^2}{\Delta x} \left( y^{n}_0 - y^{n-1}_0 \right) \right] \]

\[
\times \left[ T_0(0) + w_0(0) \left( y^{n}_0 - y^{n-1}_0 \right)^2 \right] + \frac{\Delta t}{2} b_0(y^{n}_0, n\Delta t)y^{n-1}_0 \right], \tag{20}
\]

where

\[
f_0(n\Delta t) \cong - \frac{k_0}{\Delta t} \left( y^{n}_0 - y^{n-1}_0 \right) + y_0(n\Delta t) - y_0(n\Delta t - \Delta t) \]

\[
+ \frac{3}{\Delta t} \left[ T_n + w_0 \left( y^{n}_0 - y^{n-1}_0 \right)^2 \right] \frac{y^{n}_0 - y^{n-1}_0}{\Delta x} \]

\[
- \left[ 3c_1 T_n \frac{y^{n}_0 - y^{n-1}_0}{\Delta x} + c_2 \frac{y^{n}_0 - y^{n-1}_0}{\Delta t} \right] \]

\[
+ 3c_1 w_0 \left( y^{n}_0 - y^{n-1}_0 \right)^3 \text{sign} \left( y^{n}_0 - y^{n-1}_0 \right) \]

\[
+ y_0(n\Delta t) - y_0(n\Delta t - \Delta t) \right] / \Delta t \right], \tag{21}
\]

which is obtained by applying forward approximation (16) to \( y_{x}(0,t) \) but backward approximation (17) to \( y_0(t) \) and \( \bar{y}_0(t) \). Similarly, At \( k = N_x \) or \( x = L \), we have

\[
y^{n+1}_N \cong \left[ M_L + \frac{\Delta t}{2} b_L(y^{n}_N, n\Delta t) \right]^{-1} \left[ 2M_L y^{n}_N - M_L y^{n-1}_N - M_L y_{x}(n\Delta t + \Delta t) - 2y_N(n\Delta t) + \Delta t^2 f_L(n\Delta t) \right] \]

\[
+ \frac{\Delta t^2}{\Delta x} \left( y^{n}_N - y^{n-1}_N \right) \left[ T_0(N_x \Delta x) \right] \]

\[
+ w(N_x \Delta x) \left( y^{n}_N - y^{n-1}_N \right)^2 \right] + \frac{\Delta t}{2} b_L(y^{n}_N, n\Delta t)y^{n-1}_N \right], \tag{22}
\]

where

\[
f_L(n\Delta t) \cong - \frac{3}{8} \left[ (1 - c_1) T_n \left[ k_1 \left( y^{n}_N - y^{n-1}_N \right) - \frac{y^{n}_N - y^{n-1}_N}{\Delta x} \right) \right] \]

\[
+ M_L \left( y^{n}_N - y^{n-1}_N \right) - y^{n}_N - \frac{y^{n}_N - y^{n-1}_N}{\Delta x} \right] \]

\[
+ \frac{\Delta t^2}{\Delta x} \left( y^{n}_N - y^{n-1}_N \right) \left[ T_0(N_x \Delta x) \right] \]

\[
+ 3c_1 \frac{\left( y^{n}_N - y^{n-1}_N \right)^3}{\Delta x} \text{sign} \left( y^{n}_N - y^{n-1}_N \right) \]

\[
+ 3c_1 \frac{\left( y^{n}_N - y^{n-1}_N \right)^3}{\Delta x} \left[ T_0(N_x \Delta x) \right] \]

\[
+ \frac{3}{8} \frac{\left( y^{n}_N - y^{n-1}_N \right)^3}{\Delta x} \left[ T_0(N_x \Delta x) \right] \left( y^{n}_N - y^{n-1}_N - y^{n}_N - y^{n-1}_N \right) \right] \tag{23}
\]

Equations (19), (20) and (22) can be used to calculate \( y^{n+1}_k \) (with respect to time index \( n \)) except for the case that \( n = 0 \) as \( y^{0}_k \) is not defined. Thus, equations should be developed (as the third set of equations) to compute \( y^{1}_k \), which can be done using the backward difference approximation (17). It follows from the initial conditions in (6) that

\[
y^{0}_k \approx y(k\Delta x,0), \tag{24}
\]

and therefore

\[
y^{1}_k \approx \Delta ty_{x}(k\Delta x,0) + y^{0}_k. \tag{25}
\]

In summary, we can use the finite difference approach to simulate the nonlinear string system under nonlinear robust control. Specifically, we will start with (24) and (25), use difference equations (20) and (22) to compute the time progression of boundary conditions, and use difference equation (19) to calculate both spatial and time evolution of the system response. For closed loop response, equations (21) and (23) are used; and for open-loop response, \( f_0(n\Delta t) \) and \( f_L(n\Delta t) \) are set to be zero. While the overall solution is of first order (as approximations (16) and (17) are first order approximations), we choose to use the central difference formula whenever possible to increase the simulation accuracy. Difference equations from (19) to (23) can be simplified if only first-order approximations are used.
5.2 Simulation Setup and Results

Simulations of open-loop and closed-loop responses were carried out using the following parameters and setups:

- All initial conditions in (6) are set to be zero.
- Nominal values for system parameters are: \( l = 0.5 \text{m}, T_n = 0.2 \text{N}, w_0 = 1, m_0 = 1 \text{kg/m}, M_0 = 10 \text{kg}, \) and \( M_l = 10 \text{kg}. \)
- Bounds on uncertainties are set as: \( c_1 = 0.3, c_2 = 0.1, \) and \( c_3 = 0.6 \pi / l. \)
- "Uncertainties" in the simulation are set to be
  \[
  T_0(x) = T_n + 0.3 T_n \sin \left( \frac{2 \pi}{l} x \right),
  \]
  \[
  w(x) = w_0 + 0.3 w_0 \sin \left( \frac{\pi}{l} x \right),
  \]
  \[
  m(x) = m_0 + 0.3 m_0 \cos \left( \frac{2 \pi}{l} x \right),
  \]
  \[
  b_0(p_0, t) = 9 + \cos(p_0(t)),
  \]
  and
  \[
  b_1(p_1, t) = 9 - \sin(2p_1(t)).
  \]
- In the simulation, transporter speed is set to be
  \[
  \dot{v}_b(t) = \begin{cases} 
  V_{\text{cruise}} + \Delta v_b & \text{if } 0 \leq t \leq 5 \\
  V_{\text{cruise}} + \Delta v_b & \text{if } t \geq 5
  \end{cases}
  \]
  where \( V_{\text{cruise}} = 3 \text{m/s} \) is the desired cruising speed, and \( \Delta v_b = 0.1 \sin(t + 0.15) \) is the velocity variation induced by disturbances to the transporter.
- Control gains are: \( k_0 = 2 \) and \( k_1 = 0.4267. \)
- Grid distances are \( \Delta x = 0.005 \text{m} \) and \( \Delta t = 0.001 \text{second}, \) and therefore total grid numbers are \( N_x = 101 \) and \( N_t = 27001 \) for a 27-second open-loop response, \( N_x = 101 \) and \( N_t = 50001 \) for a 50-second closed-loop response.

Open-loop responses (without control) at three different points (\( x = 0, 0.5l, l \)) are shown in figures 2, 3, and 4, respectively. In comparison, closed-loop responses of the system at points \( x = 0, 0.5l, l \) are given in figures 5, 6 and 7, respectively. Compared to the 50-second simulation for the closed loop system, the open-loop response is only simulated up to 27 seconds as it diverges soon afterwards and is not Lyapunov stable.

To demonstrate the performance of the whole string, the following spatial \( L_2 \) norm is introduced

\[
L(n) = L(n \Delta t) = \sum_{k=0}^{N_x} |Y_t(k,n) - V_{\text{cruise}}|^2,
\]
and its trajectory of time for the closed loop system is shown in figure 8. These results demonstrate the effectiveness and robustness of the proposed control in suppressing transverse vibrations.

![Figure 2: The open loop velocity at \( x = 0 \) and with respect to ground](image)

![Figure 3: The open loop velocity at \( x = 0.5l \) and with respect to ground](image)

6 Conclusion

In this paper, the vibration control problem of a nonlinear string system is considered. In order to damp out oscillation, a new robust control is designed to compensate for nonlinear uncertainties in string dynamics and in control mechanism. It is shown using the Lyapunov direct method that the proposed
control makes the closed loop system exponentially stable. Effectiveness of the control is demonstrated through nonlinear simulation and using the finite difference approach.

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References


Figure 8: The value of the spatial $L_2$ norm versus time: robust control


