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Safety Rules and Operating Procedures

1. Note the location of the Emergency Disconnect (red button near the door) to shut off power in an emergency. Note the location of the nearest telephone (map on bulletin board).
2. Students are allowed in the laboratory only when the instructor is present.
3. Open drinks and food are not allowed near the lab benches.
4. Report any broken equipment or defective parts to the lab instructor.
   Do not open, remove the cover, or attempt to repair any equipment.
5. When the lab exercise is over, all instruments, except computers, must be turned off.
   Return substitution boxes to the designated location. Your lab grade will be affected if your laboratory station is not tidy when you leave.
6. University property must not be taken from the laboratory.
7. Do not move instruments from one lab station to another lab station.
8. Do not tamper with or remove security straps, locks, or other security devices.
   Do not disable or attempt to defeat the security camera.
9. **ANYONE VIOLATING ANY RULES OR REGULATIONS MAY BE DENIED ACCESS TO THESE FACILITIES.**

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*I have read and understand these rules and procedures. I agree to abide by these rules and procedures at all times while using these facilities. I understand that failure to follow these rules and procedures will result in my immediate dismissal from the laboratory and additional disciplinary action may be taken.*
Laboratory Safety Information

Introduction

The danger of injury or death from electrical shock, fire, or explosion is present while conducting experiments in this laboratory. To work safely, it is important that you understand the prudent practices necessary to minimize the risks and what to do if there is an accident.

Electrical Shock

Avoid contact with conductors in energized electrical circuits. Electrocution has been reported at dc voltages as low as 42 volts. 100ma of current passing through the chest is usually fatal. Muscle contractions can prevent the person from moving away while being electrocuted.

Do not touch someone who is being shocked while still in contact with the electrical conductor or you may also be electrocuted. Instead, press the Emergency Disconnect (red button located near the door to the laboratory). This shuts off all power, except the lights.

Make sure your hands are dry. The resistance of dry, unbroken skin is relatively high and thus reduces the risk of shock. Skin that is broken, wet, or damp with sweat has a low resistance.

When working with an energized circuit, work with only your right hand, keeping your left hand away from all conductive material. This reduces the likelihood of an accident that results in current passing through your heart.

Be cautious of rings, watches, and necklaces. Skin beneath a ring or watch is damp, lowering the skin resistance. Shoes covering the feet are much safer than sandals.

If the victim isn’t breathing, find someone certified in CPR. Be quick! Some of the staff in the Department Office are certified in CPR. If the victim is unconscious or needs an ambulance, contact the Department Office for help or call 911. If able, the victim should go to the Student Health Services for examination and treatment.

Fire

Transistors and other components can become extremely hot and cause severe burns if touched. If resistors or other components on your proto-board catch fire, turn off the power supply and notify the instructor. If electronic instruments catch fire, press the Emergency Disconnect (red button). These small electrical fires extinguish quickly after the power is shut off. Avoid using fire extinguishers on electronic instruments.

Explosions

When using electrolytic capacitors, be careful to observe proper polarity and do not exceed the voltage rating. Electrolytic capacitors can explode and cause injury. A first aid kit is located on the wall near the door. Proceed to Student Health Services, if needed.
1 EXPERIMENT: SPECTRUM ANALYSIS

1.1 Objective:
Analyze the spectral content of a simple signal.

1.2 Equipment:
The equipment used in this experiment are:

- Oscilloscope: Rohde & Schwarz RTM 3004
- Function Generator: Tektronix AFG 3022B
- Bring a USB Flash Drive to store your waveforms.

1.3 Implementation:
A waveform representing amplitude, as a function of time, is called a time domain display. It is also possible for a waveform to represent amplitude as a function of frequency. This is called a frequency domain display. A Spectrum Analyzer is an instrument, which can display the frequency domain of a signal. However, the RTM3004 Oscilloscope has the capability of producing both time domain and frequency domain displays.

A sine wave is the simplest signal for spectral analysis. The amplitude of the sine wave can be determined on the vertical scale and the frequency can be determined on the horizontal scale.

The units of amplitude used in this experiment will be dBV, which is dB relative to 1 VRMS (0 dBV = 1 VRMS), according to the formula:

\[ \text{dBV} = 20 \log \left( \frac{V_{\text{signal}}}{V_{\text{ref}}} \right) \]  

(1.1)

where \( V_{\text{signal}} \) is the RMS voltage of the signal and \( V_{\text{ref}} = 1 \) volt RMS.

Another common unit of amplitude used for spectrum analysis is dBm, which is dB relative to 1 milliwatt, according to the formula:

\[ \text{dBm} = 10 \log \left( \frac{P_{\text{signal}}}{P_{\text{ref}}} \right) \]  

(1.2)

where \( P_{\text{signal}} \) is the power of the signal in milliwatts and \( P_{\text{ref}} = 1 \) mW.
1.4 Pre-lab Questions:

a. Calculate the amplitude in dBV of a 4 KHz, 2 volt peak-to-peak, sine wave.

b. Calculate the peak-to-peak voltage of a $-10$ dBV, 4 KHz, sine wave.

1.5 Procedure:

1.5.1 Time Domain

a. Set the Function Generator for a sine wave output with a frequency of 4 KHz. Set the amplitude to 2 volt peak-to-peak with zero DC offset.

Oscilloscope Reset:

b. Connect the Channel 1 output of the Function Generator to Channel 1 input of the Oscilloscope using a BNC to BNC cable. Turn on the Channel 1 output of the Function Generator.

c. Set the oscilloscope to the default state (PRESET).

d. Before beginning the exercise, configure the oscilloscope to use the maximum sample rate.

Switch the oscilloscope to MAX. SA. RATE mode. You can achieve that by clicking on the second upper label near C1 and change the GSa/s Acquire Mode to High Resolution after a menu appears on the right of the screen.

View the time domain waveform on the Oscilloscope:

e. Press Auto Set.

This sets the vertical, horizontal, and trigger for a stable display of the waveform.

f. Press Channel 1 button to display the menu on the right side of the screen and select an Input Impedance of 50 Ohms under the Termination label.

For this experiment, the output of the Function Generator must be terminated into a 50 Ohm load.

The time domain waveform should now be displayed.

1.5.2 Frequency Domain

View the frequency domain waveform of the input signal on the Oscilloscope:

a. Change the Horizontal Scale to display 10 cycles. You can achieve that either by rotating the second upper knob until 10 cycles appear on the screen. Or by double clicking on the top pane where the the value of seconds per division appears (the label to the left of Run). This is called the Timebase. A scroll menu containing specified durations per division appear.

Since, there are a total of 12 horizontal divisions and a total of 10 vertical divisions. For a $4 KHz$ sine wave where the center of the screen is at $0s$ time. You can find the time duration per division:
Each cycle of the sine wave is \( \frac{1}{4000} \text{s} = 0.00025 \text{s} = 0.25 \text{ms} = 250 \mu\text{s} \). Then 10 cycles will correspond to \( 10 \times 250 \mu\text{s} = 2500 \mu\text{s} \).

But the total number of horizontal divisions is 12. Then, for 10 cycles to appear we need the time duration per division (i.e., Timebase) to be \( \frac{2500}{12} = 208.33 \mu\text{s} / \text{div} \). But the available durations per division are specified. So pick the value appearing in the scroll menu just above the calculated value, that is 500\( \mu\text{s} \). Or click on the menu button and adjust the value to the calculated one 208.33\( \mu\text{s} / \text{div} \).

b. Press on the FFT icon touch button on the right top pane.

The screen is split into two panes; see Fig. 1.1. The upper pane (2) displays the previously defined voltage-time trace for the measured signal and the settings for the horizontal system. That is, the waveform in the time domain. The zoom and position information is displayed in between the two panes (3). Only the signal segment that is displayed on the monitor and bound by the two vertical white lines is used for the FFT.

The lower pane (4) shows the result of the FFT analysis, that is, it will display the spectrum of the signal.
The FFT displays the frequency domain by expressing the input signal as a combination of simple signals and produces a waveform to display the amplitude verses frequency of the component frequencies of the input signal.

c. Now, either
i) Double press the FFT button under the Analysis grouping and a menu will appear on the right of the screen (Fig. 1.2),
ii) or by clicking on the FFT label on the bottom pane, a menu will appear (Fig. 1.3), then Press on Menu. The right menu will appear (Fig. 1.2).

On this right pane menu (Fig. 1.2), these should be selected, if not already set:

- FFT Source: C1 (Channel 1)
- FFT Window: Hanning
- Vertical Scale: dBV RMS
- Automatic RBW: off

d. First, adjust the time duration per division (timebase) on the top pane from $833.33\mu s/\text{div}$ to $200\text{ms/\text{div}}$ and then select a maximum signal segment for the FFT.
Span (4 in Fig. 1.4) shows the size of the currently displayed frequency range. The span, time-
base and selected segment are directly linked to one another. As the selection for the timebase
and the range in the time domain displayed between the two vertical lines increases, so too
decreases the frequency band that can be set. Center defines the frequency at the center of
the segment displayed on the monitor. The minimum step size $\Delta f$ can be set indirectly via the
number of points.

e. Set now the value of the RBW (5 in Fig. 1.4) to the maximum (i.e. by clicking on RBW on the bar between
the two screen and adjusting the value to the maximum allowed).

f. On the bar between the two panes that display time and frequency domains respectively, set Start (1 in
Fig. 1.4) value to 0, you can achieve that by clicking on the Start label where calculator form menu appears
and you can set the value. This will be the value of the minimum frequency displayed.

g. Similarly, adjust the Stop (2 in Fig. 1.4) value to a value above 4KHz, let's say 8KHz. N.B. Usually
you need this value of 8KHz to fall between the minimum value and maximum value allowed for the Stop
value. Therefore, you need to readjust the time duration per division (timebase) so that is satisfied. Adjust
the center frequency to 4KHz you can set this value through clicking on Center in the middle bar. See the
blue frame comments.

Using the Cursors:

First, for a better view of the frequency domain screen. Move the double arrow bar of the lower screen
upwards so that the frequency domain is the only display on the oscilloscope. There are a variety of ways
to measure the amplitude. One method is to use the Cursors.

a. Press the Cursors button twice on the front panel to activate the cursors. At the first click a bar will appear
showing the values at the used cursors. After the second click a right menu will appear. On the Type, set to
Vertical & Horizontal, so that both types of cursors appear on the screen. Set the Source to FFT: Spectrum,
so that the frequency domain cursors appear.

Cursors 1 and 2 are the horizontal cursors.
Cursors 3 and 4 are the vertical cursors.
Click on number 1 on the cursor line and move it to the desired point on the spectrum. Its peak value in
our case. The amplitude measured by the Cursor 1 is now displayed on the bar near label L1 and the
corresponding frequency value near label f1.

b. Does the amplitude agree with the value that you calculated in the Preparation?
c. Save the Screen Image to your USB Flash Drive to include in your lab report by clicking on the camera icon in the Action section of the oscilloscope buttons.
d. Change the frequency of the Function Generator to 2 KHz.
Does the spectrum display accurately represent the input signal?
Does it appear on the frequency domain display?
Can you adjust any of the labels in the middle bar so that the peak value appears?
Are you able to adjust accurately the setup so that the center frequency is at $2KHz$.
e. Change the frequency of the Function Generator to 1 KHz.
Does the frequency domain waveform confirm that the signal is 1 KHz?
Are you able to answer the questions for part d.

**Taking a Measurement:**

a. Decrease the amplitude of the Function Generator output to $-10$ dBV as measured in the frequency domain display. Now, switch to the time domain display. Press on the measure button. A menu will appear on the right of the screen. Set the Measure Place to 1. Turn the Measure 1 bar to on. Choose the Type to the type of measurement (peak to peak). Read the peak-to-peak voltage measurement that will appear on the bottom left of the screen above the channels bar. Does this measurement agree with the value that you calculated in the Preparation?

**Using the Zoom Feature:**

a. Turn the time domain on again by pressing the CH1 button. Turn the Zoom on by pressing the Zoom button. Press on the part of the signal you want to zoom in. By using the two fingers motion that you use on the i-pad for zooming, double-tap with two fingers to zoom in and back out so that you can adjust the signal part you want to inspect to the right resolution.

**1.6 Report:**

In your report, describe what you have learned in this experiment. Compare your experimental measurements with the theoretical calculations. Remember to insert the picture that you saved as part of your report. Write all conclusions. **Follow the instructions listed in the Appendix regarding outline and required analysis.**
1.7 Hints & Comments

1.7.1 Procedure

1.5.1 Time Domain: Students should not mistakenly connect the input from the function generator directly to the oscilloscope. Instead, they should construct a circuit with a 50 Ω resistor and then connect the output of that resistor to the oscilloscope. A figure can help resolve this misunderstanding (see Fig. 1.5 above).

Students must not forget to adjust the input impedance of the oscilloscope to 50 Ω instead of 1 MΩ.
2. EXPERIMENT: MATLAB

2.1 Objective:

The purpose of this experiment is to be able to implement a signal analysis problem through simulation by either using MATLAB.

2.2 Equipment:

The equipment used in this experiment are:

- Desktop or laptop computer
- MATLAB software

2.3 Simulation

MATLAB

MATLAB is a programming language for technical computing. MATLAB is case sensitive. A variable in MATLAB is an array. An array has dimensions $m \times n$, $m$ is the number of rows and $n$ is the number of columns. If $m = n = 1$, the variable is a scalar. If $m = 1$ and $n > 1$, the variable is a row vector. If $n = 1$ and $m > 1$, the variable is a column vector. If both $m$ and $n$ are greater than one, the variable is a matrix, and if $m = n$, then the variable is a square matrix. The coefficients of an array can be real or complex numbers. Please refer to Slide 2.1 for some examples on data types and data accessing.

Slide 2.1

A previously typed command can be recalled with the up-arrow key. When the command is displayed at the command prompt, it can be modified if needed and executed.
See Slides 2.2 and 2.3 for defining some special matrices (all-zero, all-one, and identity matrices) and some basic commands. See Slide 2.4 for some helpful commands.

For predefined functions and constants see Table 2.1 and Table 2.2.

For plotting functions see Slide 2.5.

Compact representation of a vector: for example instead of $A = \begin{bmatrix} 1 & 3 & 5 & 9 & 11 & 13 \end{bmatrix}$ we can say $A = 1 : 2 : 13$.

Another example is $B = 0 : \pi/100 : 2 \pi$ is a vector that starts at 0 takes steps or increments of $\frac{\pi}{100}$ stops when $2\pi$ is reached. If you omit the increment MATLAB automatically increments by 1.

For **element-wise operations** on array see Slide 2.6, and Slide 2.7 for some examples.

For inequalities and equations see Slide 2.8. And for logical operators see Slide 2.9.

For plotting use Slide 2.10, 2.11 and Fig. 2.1.

For multiple plots check Slide 2.12.
In MATLAB, it becomes

\[
\frac{1}{2+3^2} + \frac{4}{5} \times \frac{6}{7}
\]

```matlab
>> 1/(2+3^2)+4/5*6/7
ans =
0.7766
```

- clear all
- clc
- a=2; b=5; d=[6 7];
- clr clears your workspace and command window, so you can start fresh.
- clr performs: clear all; close all; clc; clr is a quick way to "reset" Matlab.

- help command
- doc command
- lookfor inverse: look for inverse searches for the character vector inverse in the first comment line (the H1 line) of the help text in all MATLAB® program files found on the search path
Table 2.1: Elementary functions

<table>
<thead>
<tr>
<th>Function</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>cos(x)</code></td>
<td>Cosine</td>
</tr>
<tr>
<td><code>sin(x)</code></td>
<td>Sine</td>
</tr>
<tr>
<td><code>tan(x)</code></td>
<td>Tangent</td>
</tr>
<tr>
<td><code>acos(x)</code></td>
<td>Arc cosine</td>
</tr>
<tr>
<td><code>asin(x)</code></td>
<td>Arc sine</td>
</tr>
<tr>
<td><code>atan(x)</code></td>
<td>Arc tangent</td>
</tr>
<tr>
<td><code>exp(x)</code></td>
<td>Exponential</td>
</tr>
<tr>
<td><code>sqrt(x)</code></td>
<td>Square root</td>
</tr>
<tr>
<td><code>log(x)</code></td>
<td>Natural logarithm</td>
</tr>
<tr>
<td><code>log10(x)</code></td>
<td>Common logarithm</td>
</tr>
<tr>
<td><code>abs(x)</code></td>
<td>Absolute value</td>
</tr>
<tr>
<td><code>sign(x)</code></td>
<td>Signum function</td>
</tr>
<tr>
<td><code>max(x)</code></td>
<td>Maximum value</td>
</tr>
<tr>
<td><code>min(x)</code></td>
<td>Minimum value</td>
</tr>
<tr>
<td><code>ceil(x)</code></td>
<td>Round towards +∞</td>
</tr>
<tr>
<td><code>floor(x)</code></td>
<td>Round towards −∞</td>
</tr>
<tr>
<td><code>round(x)</code></td>
<td>Round to nearest integer</td>
</tr>
<tr>
<td><code>rem(x)</code></td>
<td>Remainder after division</td>
</tr>
<tr>
<td><code>angle(x)</code></td>
<td>Phase angle</td>
</tr>
<tr>
<td><code>conj(x)</code></td>
<td>Complex conjugate</td>
</tr>
</tbody>
</table>

Table 2.2: Predefined constant values

<table>
<thead>
<tr>
<th>Constant</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><code>pi</code></td>
<td>The π number, π = 3.14159…</td>
</tr>
<tr>
<td><code>i, j</code></td>
<td>The imaginary unit i, √−1</td>
</tr>
<tr>
<td><code>Inf</code></td>
<td>The infinity, ∞</td>
</tr>
<tr>
<td><code>NaN</code></td>
<td>Not a number</td>
</tr>
</tbody>
</table>

Plot

The basic MATLAB graphing procedure, for example in 2D, is to take a vector of x-coordinates, \( x = (x_1, \ldots, x_N) \), and a vector of y-coordinates, \( y = (y_1, \ldots, y_N) \), locate the points \((x_i, y_i)\), with \(i = 1, 2, \ldots, n\) and then join them by straight lines. You need to prepare \(x\) and \(y\) in an identical array form; namely, \(x\) and \(y\) are both row arrays or column arrays of the same length.

The MATLAB command to plot a graph is `plot(x,y)`. The vectors \(x = (1, 2, 3, 4, 5, 6)\) and \(y = (3, -1, 2, 4, 5, 1)\) produce the picture shown in Figure 2.1.

```
>> x = [1 2 3 4 5 6];
>> y = [3 -1 2 4 5 1];
>> plot(x,y)
```
Slide 2.6

\[
\begin{array}{c|c}
.* & \text{Element-by-element multiplication} \\
./ & \text{Element-by-element division} \\
.^{} & \text{Element-by-element exponentiation}
\end{array}
\]

Table 3.1: Array operators

\[
A = \begin{bmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{bmatrix}, \quad B = \begin{bmatrix}
10 & 20 & 30 \\
40 & 50 & 60 \\
70 & 80 & 90
\end{bmatrix}
\]

we have,

\[
\begin{align*}
\text{>> } & C = A.*B \\
C &= \\
& \begin{bmatrix}
10 & 40 & 90 \\
160 & 250 & 360 \\
490 & 640 & 810
\end{bmatrix}
\end{align*}
\]

\[
\begin{align*}
\text{>> } & A.^{2} \\
\text{ans } &= \\
& \begin{bmatrix}
1 & 4 & 9 \\
16 & 25 & 36 \\
49 & 64 & 81
\end{bmatrix}
\end{align*}
\]

Slide 2.7

**Inequalities and Equalities**
- less than \(<\)
- less than or equal \(\leq\)
- greater \(>\)
- greater than or equal \(\geq\)
- equal to \(==\)
- not equal to \(\sim=\)
LOGICAL OPERATORS

- not ~
- and &
- or |

Slide 2.9

<table>
<thead>
<tr>
<th>SYMBOL</th>
<th>COLOR</th>
<th>SYMBOL</th>
<th>LINE STYLE</th>
<th>SYMBOL</th>
<th>MARKER</th>
</tr>
</thead>
<tbody>
<tr>
<td>k</td>
<td>Black</td>
<td>—</td>
<td>Solid</td>
<td>+</td>
<td>Plus sign</td>
</tr>
<tr>
<td>r</td>
<td>Red</td>
<td>——</td>
<td>Dashed</td>
<td>o</td>
<td>Circle</td>
</tr>
<tr>
<td>b</td>
<td>Blue</td>
<td>:</td>
<td>Dotted</td>
<td>*</td>
<td>Asterisk</td>
</tr>
<tr>
<td>g</td>
<td>Green</td>
<td>.</td>
<td>Dash-dot</td>
<td>.</td>
<td>Point</td>
</tr>
<tr>
<td>c</td>
<td>Cyan</td>
<td>none</td>
<td>No line</td>
<td>x</td>
<td>Cross</td>
</tr>
<tr>
<td>m</td>
<td>Magenta</td>
<td></td>
<td></td>
<td>s</td>
<td>Square</td>
</tr>
<tr>
<td>y</td>
<td>Yellow</td>
<td></td>
<td></td>
<td>d</td>
<td>Diamond</td>
</tr>
</tbody>
</table>

Slide 2.10: Attributes on the plot

- t=-20:0.1:20;
- g=sin(t)./t;
- plot(t,g)
- xlabel('time')
- ylabel('g(t)')
- title('plot of g(t) versus t')
- % default color of a graph is blue. you can change it:
- plot(g,t,'r')
- plot(g,t,'g*')
- plot(g,t,'g--o')

Slide 2.11
Fig. 2.1: Plot of $\text{sinc}$ function

- multiple plots

```matlab
>> x = 0:pi/100:2*pi;
>> y1 = 2*cos(x);
>> y2 = cos(x);
>> y3 = 0.5*cos(x);
>> plot(x,y1,'--',x,y2,'-',x,y3,:')
>> xlabel('0 \leq x \leq 2\pi')
>> ylabel('Cosine functions')
>> legend('2*cos(x)','cos(x)','0.5*cos(x)')
```
For plotting discrete time signals use `stem` (see Fig. 2.2).

For using complex numbers check Slide 2.13.

For using loops and conditional statements check Slide 2.14 and example in Slide 2.15.

In MATLAB you can save programs in a file and execute them from the command line. The file is called an **m-file**, and can be assigned a name. You can execute the programs in the file by entering the name of the file on the command line. You can save all our scripts on an m-file. It is a good idea to always start an m-file with **clear all**. You can write a function and call it inside your m-file

```
function [outputs] = function_name(inputs)
```

Save the example in Slide 2.16 as `myfactorial.m`
Loops and Conditional Statements

- for----end
- while----end
- if----else----end
- If----elseif----elseif----end

If expression
  statements
elseif expression
  statements
else
  statements
end

Slide 2.14

```matlab
for i=-5:5
  if i>0
    disp('positive number');
  elseif i==0
    disp('zero');
  elseif i<0
    disp('negative number');
  end
end
```

Slide 2.15

```matlab
function m=myfactorial(n)
m=1;
for i=1:n
  m=m*i;
end
```

Slide 2.16
2.4 MATLAB Implementation

Problem 1
Let

\[ x_1(t) = t^2 - 2t + 3 \quad -10 \leq t \leq 10 \]
\[ x_2(t) = 4\cos(2\pi t - \frac{\pi}{8}) + 3\sin2\pi t \quad -2 \leq t \leq 2 \]
\[ x_3(t) = \text{sinc}(t) = \frac{\sin(t)}{t} \]
\[ x_4(t) = e^{-at} \]

a. Plot \( x_1(t) \) and \( x_2(t) \).

b. Plot \( x_3(t) \) for \(-2\pi \leq t \leq 2\pi \)

c. Plot \( e^{-at} \) for \( a = 0.1, 1, 3 \) and \( 0 \leq t \leq 10 \) on one plot. Use different colors or markers for each plot.

Problem 2
a. For the functions in Problem 1, plot their discrete versions using \textit{stem} where \( n \) varies on the integers.

\[ x_1(n) = n^2 - 2n + 3 \quad -10 \leq n \leq 10 \]
\[ x_2(n) = 4\cos(2\pi n - \frac{\pi}{8}) + 3\sin2\pi tn \quad -2 \leq n \leq 2 \]
\[ x_3(n) = \text{sinc}(n) = \frac{\sin(n)}{n} \quad -5 \leq n \leq 5 \]
\[ x_4(n) = e^{-an} \quad 0 \leq n \leq 10 \quad \text{for} \ a = 0.1, 1, 3 \]

For each of the above functions compare the graphs of Problem 1 and their corresponding samples in Problem 2 by plotting both on the same plot.

Problem 3
For functions \( x_1(t), x_2(t), x_3(t) \), write a program that plots a graph of a function \( y(t) \) for \(-4 < t < 4\) such that

\[
 y(t) = \begin{cases} 
 x_1(t) & -4 < t < -2 \\
 x_2(t) & -2 < t < 2 \\
 x_3(t) & 2 < t < 4 
\end{cases}
\]

a. Write the above program using a \textit{for} loop and \textit{conditional} statements.
b. Write the above program using vector evaluations. i.e., \textit{plot}(t_1, x_1, t_2, x_2, t_3, x_3)
3 EXPERIMENT: Periodic Signal Spectra

3.1 Objective:

To understand the relationships between time waveforms and frequency spectra.

3.2 Equipment:

The equipment used in this experiment are:

- Oscilloscope: Rohde & Schwarz RTM 3004
- Function Generator: Tektronix AFG 3022B
- Bring a USB Flash Drive to store your waveforms.

3.3 Theory:

3.3.1 Motivation

Consider for a moment the special class of systems that are called linear time-invariant systems. When a signal is applied to this system, the output can be expressed in terms of linear differential equations with constant coefficients. An example of such a system is shown in Figure 3.1. The differential equation that describes the input/output relationship in such a system is given below:

\[
\frac{d}{dt} v_o + \frac{R}{L} v_o = \frac{R}{L} v_i
\]

(3.1)

For a finite input \(v_i(t)\) we expect to solve this differential equation to find the corresponding output \(v_o(t)\). The difficulty involved in this task is that the input has to be a very simple signal. One way to deal with this complication is to express the input as a linear combination of simpler inputs. The choice of simpler inputs will affect the difficulty that we may encounter in solving the problem.

There is an intriguing possibility regarding the simpler input signals. Let us, for a moment, assume that we choose as an input a function that repeats itself under the operation of differentiation. We can show that such a function at the input will yield the same function as an output multiplied by some algebraic polynomial involving the parameters of the function and the parameters of the differential equation.

![Fig. 3.1: A Simple linear Time-invariant System](image-url)
A function that demonstrates this behavior, that is to repeat itself under the operation of differentiation, is the function $e^{st}$, where $s = \sigma + j\omega$ is a complex number. Since some of the signals that we are interested in are energy-type signals that may exist for positive or negative time it is wise to choose $\sigma = 0$. This discussion indicates that if the input signal $v_i(t)$ in equation (3.1) is of the form $e^{j\omega t}$, then we can readily find the output $v_o(t)$ of the system. Furthermore, if we can express the input signal $v_i(t)$ as a linear combination of complex exponential signals (of the form $e^{j\omega t}$), we can still find, without difficulty, the output $v_o(t)$ of the system (the reason being that the system in (3.1) is linear, and as a result if the input $v_i(t)$ is a linear combination of complex exponentials, then the output $v_o(t)$ can be expressed as a linear combination of the outputs of the complex exponentials involved in the input). One version of the Fourier series expresses an arbitrary signal (like $v_i(t)$) as a linear combination of complex exponentials. This is the first reason that the Fourier series representation of a signal is so useful.

A close relative of a complex exponential $e^{j\omega t}$ are the signals $\cos \omega t$ and $\sin \omega t$ (remember Euler’s identity: $e^{j\omega t} = \cos \omega t + j\sin \omega t$). Consider now the following set of familiar signals.

\[
\begin{align*}
  f_1(t) &= a \sin \omega_0 t \\
  f_2(t) &= a \sin 2\omega_0 t \\
  f_3(t) &= a \sin 3\omega_0 t \\
  \vdots \\
  f_n(t) &= a \sin n\omega_0 t \\
\end{align*}
\]  

(3.2)

In Figure 3.2 we show a plot of $f_1(t)$ and $f_2(t)$. More generally, we can say that as $n$ increases, the rate of variation of $f_n(t)$ increases as well. One way to quantify the above qualitative statements is to take the derivative of $f_n(t)$ with respect to time. In particular,

\[
d \frac{df(t)}{dt} = an\omega_0 t
\]  

(3.3)

From the above equation we observe that as $n$ increases the maximum rate of variation of $f_n(t)$ with respect $t$ time increases. We can represent each one of the $f_n(t)$ functions in a little bit different way than the one used in Fig. 3.2. That is, we can represent these functions by plotting the maximum amplitude of $f_n(t)$ (i.e., $a$) versus the angular frequency of the sinusoid (i.e., $n\omega_0$). The product of these two values gives you the maximum rate of variation for the signal under consideration. In Fig. 3.3 we are representing the functions $f_1(t), f_2(t),$ and $f_3(t)$ following the aforementioned convention. Obviously, larger values for the location of the plotted amplitudes imply faster varying time signals; also the larger the amplitudes plotted at a particular location the faster the corresponding signals vary with respect to time.

Now consider a simple example of an amplitude-modulated signal.

\[
f(t) = A(1 + \cos \omega_m t)\cos \omega_0 t \quad \omega_m << \omega_0
\]  

(3.4)

In Fig. 3.4 we plot the aforementioned signal $f(t)$. As we can see from the figure, the amplitude varies slowly between $0$ and $2A$. Its rate of variation is given by $\omega_m$, the modulating frequency; $\omega_0$ is referred to as the carrier frequency. The amplitude variations form the envelope of the complete signal, and represent any information being transmitted. Note that

\[
A(1 + \cos \omega_m t)\cos \omega_0 t = Acos\omega_0 t + \frac{A}{2}[\cos(\omega_0 - \omega_m)t - \cos(\omega_0 + \omega_m)t]
\]  

(3.5)
Fig. 3.2: Plots of $f_1(t)$ and $f_2(t)$

Fig. 3.3: Representation of $f_1(t)$, $f_2(t)$, and $f_3(t)$

Fig. 3.4: Plot of an amplitude-modulated signal $f(t)$
The representation of the aforementioned signal, using the new conventions, is illustrated in Fig. 3.5. As we observe from Fig. 3.5, larger carrier frequency corresponds to a faster varying signal; also larger modulating frequency results in a faster varying signal.

The previously discussed signals vary in some sinusoidal fashion. They do not carry any information but we have used them to focus on an alternative representation of the signal, that conveys to us the information of how fast the signal changes with respect to time. Another variation of the Fourier series representation of a signal represents the signal as a linear combination of sines and cosines of appropriate frequencies. As a result, the Fourier series representation of the signal, which is an extension of the alternative representation of the simple signals that we discussed above, gives us an idea of how fast the signal changes with respect to time. This is the second reason that the Fourier series representation of a signal is so useful.

### 3.3.2 Fourier Series

Let our space \( S \) of interest be the set of piece-wise continuous real time signals, defined over an interval \((0, T)\). Let us also consider the set of signals

\[
\phi_{10}(t) = \frac{1}{\sqrt{T}}
\]

\[
\phi_{1n}(t) = \frac{2}{\sqrt{T}} \cos n\omega_0 t \quad n = 1, 2, ...
\]

\[
\phi_{2n}(t) = \frac{2}{\sqrt{T}} \sin n\omega_0 t \quad n = 1, 2, ...
\]

where \( \omega_0 = \frac{2\pi}{T} \). The aforementioned signals (see equations (3.6)) are orthonormal. They also constitute a complete set of functions. These two properties allow us to state that an arbitrary signal in \( S \) can be expressed as a linear combination of the signals in equations (3.6), such that

\[
f(t) = C_{10}\phi_{10} + \sum_{n=1}^{\infty} C_{1n}\phi_{1n} + \sum_{n=1}^{\infty} C_{2n}\phi_{2n}
\]

and the coefficients in the linear expansion above can be computed through the following equations:

\[
C_{10} = \frac{1}{\sqrt{T}} \int_{0}^{T} f(t) dt
\]

\[
C_{1n} = \frac{2}{\sqrt{T}} \int_{0}^{T} f(t) \cos n\omega_0 t dt \quad n = 1, 2, ...
\]
\[ C_{2n} = \frac{2}{\sqrt{T}} \int_0^T f(t) \sin n\omega_0 t \, dt \quad n = 1, 2, \ldots \] (3.8c)

The above expansion, due to Fourier, is referred to as the Trigonometric Fourier Series expansion of a signal \( f(t) \). A more common form for the trigonometric Fourier series expansion of a signal is given below.

\[ f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t \] (3.9)

where

\[ a_0 = \frac{1}{T} \int_0^T f(t) \, dt \] (3.10a)

\[ a_n = \frac{2}{T} \int_0^T f(t) \cos n\omega_0 t \, dt \quad n = 1, 2, \ldots \] (3.10b)

\[ b_n = \frac{2}{T} \int_0^T f(t) \sin n\omega_0 t \, dt \quad n = 1, 2, \ldots \] (3.10c)

By a simple trigonometric manipulation we get that

\[ f(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \cos (n\omega_0 t + \theta_n) \quad \theta_n = \arctan \left( \frac{b_n}{a_n} \right) \] (3.11)

The above equation puts \( f(t) \) in the desired form. In other words, we have now expressed \( f(t) \) as a linear combination of sinusoids, and as a result we can represent this signal \( f(t) \) by plotting the amplitude of each one of its sinusoids at the corresponding angular frequency location. This will give rise to the amplitude-frequency plot of \( f(t) \). Observe though that in equation (3.11) each sinusoid involved in the expansion of \( f(t) \) has a phase associated with it. Hence, in order to completely describe the signal \( f(t) \) we need to draw another plot that provides the information of the phase associated with each sinusoid. This plot is denoted as the phase-frequency plot and it corresponds to a plot of \( \theta_n \) at the location \( \omega_n \).

It is worth mentioning that the aforementioned expansion of \( f(t) \) in terms of sinusoidal signals of frequencies that are integer multiples of the frequency \( \omega_0 \) is often referred to as the expansion of \( f(t) \) in terms of its harmonic components. The frequency \( \omega_0 \) is called the fundamental frequency, or first harmonic, while multiples of the first harmonic frequency are referred to as the second harmonic, third harmonic, and so on. The corresponding signals in the expansion are named first harmonic component, second harmonic component, third harmonic component, and so on. The component \( a_0 \) in the above expansion is called the DC component of the signal \( f(t) \).

Since we shall frequently be interested in the amplitudes, \( \sqrt{a_n^2 + b_n^2} \)'s, and phases, \( \theta_n \)'s, of our signal \( f(t) \) it would be simpler to obtain these directly from \( f(t) \), rather than by first finding \( a_n \) and \( b_n \). In order to achieve this, let us consider the functions

\[ \phi_n(t) = \frac{1}{\sqrt{T}} e^{jn\omega_0 t} \] (3.12)

where \( \omega_0 = \frac{2\pi}{T} \). The above set of functions is a complete, orthonormal set. Therefore, any function \( f(t) \), defined over an interval of length \( T \), can be expanded as a linear combination of the \( \phi_n \)'s. In particular,

\[ f(t) = \sum_{n=-\infty}^{\infty} C_n \phi_n(t) = \sum_{n=-\infty}^{\infty} C_n \frac{1}{\sqrt{T}} e^{jn\omega_0 t} \] (3.13)

The coefficients \( C_n \) in the above expansion are chosen according to the rules specified by the Orthogonality Theorem. That is,

\[ C_n = \frac{1}{\sqrt{T}} \int_0^T f(t) \frac{1}{\sqrt{T}} e^{jn\omega_0 t} \, dt \] (3.14)
The above equation is denoted as the **Exponential Fourier Series** expansion of the signal \( f(t) \). Actually, in most instances, the exponential Fourier series expansion of a signal is provided by the following equation:

\[
f(t) = \sum_{n=-\infty}^{\infty} F_n e^{j\omega_0 t}
\]

where

\[
F_n = \frac{1}{T} \int_{0}^{T} f(t) e^{-j\omega_0 t} dt
\]

Our original goal has now been accomplished, since we can now derive the amplitude - frequency plot of \( f(t) \) by plotting the magnitude of \( F_n \), with respect to frequency, and we can also derive the phase - frequency plot of \( f(t) \) by plotting the phase of \( F_n \) with respect to frequency. Note that, in general, the \( F_n \)'s are complex numbers and a complex number \( F_n \), can be written as the product of its magnitude (\( |F_n| \)) times \( e^{j\angle F_n} \). One can show that the coefficients of the exponential Fourier series expansion of \( f(t) \) and the coefficients of the trigonometric Fourier series expansion of \( f(t) \) are related as follows:

\[
F_0 = a_0
\]

\[
F_n = \frac{1}{2} (a_n - j b_n) \quad 1 \leq n < \infty.
\]

\[
F_{-n} = \frac{1}{2} (a_n + j b_n) \quad 1 \leq n < \infty.
\]

The following are some comments:

- The exponential Fourier series and the trigonometric Fourier series expansion were introduced for a signal \( f(t) \) defined over the interval \((0, T)\). Actually, the formulas provided are valid for any signal \( f(t) \) defined over any interval of length \( T \) (the starting and the ending points of the interval are immaterial, as far as the length of the interval is equal to \( T \)).

- We expanded so far a signal \( f(t) \), defined over an interval of length \( T \), in terms of a linear combination of sinusoids (Trigonometric Fourier Series expansion), or complex exponentials (Exponential Fourier Series expansion). It is worth noting, that the trigonometric or exponential Fourier series expansion of a signal defined over an interval \( T \) is a periodic signal of period \( T \); that is, it repeats itself with period \( T \). Hence, if the signal of interest \( f(t) \) is not periodic the trigonometric or exponential Fourier series expansion of the signal is only valid for the interval over which the signal is defined.

- Due to the periodicity nature of the Fourier series expansion, Fourier series is primarily used to represent signals of periodic nature. Signals of aperiodic nature can also be represented as a "sum" of sinusoids or complex exponentials, but this "sum" is in reality an integral and it is designated by the name Fourier Transform. The Fourier transform of an aperiodic signal will be introduced later as an extension of the Fourier series of a periodic signal.

**Example 1**

(a) Evaluate the trigonometric Fourier series expansion of a signal, which is depicted in Fig. 3.6.

The interval of interest is \((-\pi/2, \pi/2)\). Hence, \( T = \pi \) and as a result \( \omega_0 = \frac{2\pi}{T} = 2 \). The trigonometric Fourier series of a signal \( f(t) \) is therefore of the form

\[
f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2nt + \sum_{n=1}^{\infty} b_n \sin 2nt
\]
Fig. 3.6: Plot of \( f(t) = \cos t \) for \( [-\frac{\pi}{2}, \frac{\pi}{2}] \) and zero elsewhere

\[
a_0 = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) dt = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t dt = \frac{2}{\pi}
\]

\[
a_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos 2nt dt = \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \cos 2nt dt = \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\cos(2n-1)t + \cos(2n+1)t] dt
\]

\[
= \frac{2}{\pi} \left\{ (\frac{-1}{2n-1})^{n+1} + (\frac{-1}{2n+1})^{n+1} \right\}
\]

\[
b_n = \frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin 2nt dt = \frac{2}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \sin 2nt dt = 0
\]

The last equation is a result of the fact that \( f(t) \) is even and \( \sin 2nt \) is odd. Consequently, \( f(t)\sin 2nt \) is odd, and whenever an odd function is integrated over an interval which is symmetric around zero, the result of the integration is zero. Based on the above equations we can write:

\[
f(t) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \left\{ \frac{2}{\pi} \left[ (\frac{-1}{2n-1})^{n+1} + (\frac{-1}{2n+1})^{n+1} \right] \right\} \cos 2nt
\]
Also, if we write out a couple of terms from the above equation we get:

\[ f(t) = \frac{2}{\pi} + \left[ 1 + \frac{2}{3}\cos 2t - \frac{2}{15}\cos 4t + \frac{2}{35}\cos 6t + \ldots \right] \]  
(3.23)

(b) Find the exponential Fourier series of the signal \( f(t) \) in Fig. 3.6.

We can proceed via different paths. We can either find, directly, the exponential Fourier series expansion of \( f(t) \) by applying the pertinent formulas, or we can use the trigonometric Fourier series expansion already derived to generate the exponential Fourier series expansion. We choose the latter approach, because it is easier. Recall that

\[ f(t) = \frac{2}{\pi} + \sum_{n=1}^{\infty} a_n \cos 2nt \]  
(3.24)

where

\[ a_n = \frac{2}{\pi} \left[ \frac{(-1)^n+1}{2n-1} + \frac{(-1)^{n+1}}{2n+1} \right] \]  
(3.25)

Due to Euler’s identity we can write

\[ \cos 2nt = \frac{e^{j2nt} + e^{-j2nt}}{2} \]  
(3.26)

Using Euler’s identity in equation (3.26) we get

\[ f(t) = \frac{2}{\pi} + \sum_{n=1}^{\infty} \frac{a_n}{2} e^{j2nt} + \sum_{n=1}^{\infty} \frac{a_n}{2} e^{-j2nt} \]  
(3.27)

Notice also that the exponential Fourier series expansion of the signal \( f(t) \) has the following form

\[ f(t) = \sum_{n=-\infty}^{\infty} F_n e^{j2nt} = F_0 + \sum_{n=1}^{\infty} F_n e^{j2nt} + \sum_{n=1}^{\infty} F_{-n} e^{-j2nt} \]  
(3.28)

Comparing, one by one, the terms of equations (3.27) and (3.28) we deduce the following identities:

\[ F_0 = a_0 \]  
(3.29a)

\[ F_n = \frac{a_n}{2} \]  
(3.29b)

\[ F_{-n} = \frac{a_n}{2} \]  
(3.29c)

In particular,

\[ f(t) = \frac{2}{\pi} + \frac{2}{3\pi} e^{j2t} + \frac{2}{3\pi} e^{-j2t} - \frac{2}{15\pi} e^{j4t} - \frac{2}{15\pi} e^{-j4t} \]  
(3.30)
Based on equations (3.29) we can plot the amplitude - frequency, and the phase - frequency plots for the signal $f(t)$. In the case where the coefficients $F_n$ are real we can combine these two plots into one plot; this plot is denoted as the **line spectrum** of $f(t)$. The line spectrum of a signal $f(t)$ corresponds to the plot of $F_n$'s with respect to frequency. The line spectrum of the signal $f(t)$ in this example is depicted in Fig. 3.7.

**Example 2**

Evaluate the trigonometric Fourier series expansion of the periodic signal which is depicted in Fig. 3.9.

Let us concentrate on one period of the signal, that is the interval $(-\pi/2, \pi/2)$. Knowing that $T = \pi$ we conclude that $\omega_0 = \frac{2\pi}{T} = 2$. The trigonometric Fourier series of a signal $f(t)$ is therefore of the form

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos 2nt + \sum_{n=1}^{\infty} b_n \sin 2nt$$

(3.31)
where

\[
a_0 = \frac{1}{T} \int_{-T/2}^{T/2} f(t) dt = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} f(t) dt \tag{3.32}
\]

\[
a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos 2nt dt
= \frac{2}{\pi} \int_{-\pi/2}^{\pi/2} (\cos 2nt dt + \frac{2}{\pi} \int_{-\pi/4}^{\pi/4} (\cos 2nt dt + \frac{2}{\pi} \int_{\pi/4}^{\pi/2} (-1) \cos 2nt dt
= \frac{2 \sin (\frac{n\pi}{T})}{n\pi} \tag{3.33}
\]

\[
b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin 2nt dt
= 0 \tag{3.34}
\]

The last equation is a result of the fact that \( f(t) \) is even and \( \sin 2nt \) is odd. Consequently, \( f(t) \sin 2nt \) is odd, and whenever an odd function is integrated over an interval which is symmetric around zero, the result of integration is zero. Based on the above equations we can write:

\[
f(t) = \sum_{n=1}^{\infty} \left[ \frac{2 \sin (\frac{n\pi}{T})}{n\pi} \right] \cos 2nt \tag{3.35}
\]

Also, if we write out a couple of terms from the above equation we get:

\[
f(t) = \frac{4}{\pi} \left[ \cos 2t - \frac{1}{3} \cos 6t - \frac{1}{5} \cos 10t + \frac{1}{7} \cos 14t + \ldots \right] \tag{3.36}
\]

### 3.3.3 Power - Parseval’s Relation

The power \( P_f \) of a real signal \( f(t) \) is defined as follows:

\[
P_f = \lim_{T_0 \to \infty} \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} f^2(t) dt \tag{3.37}
\]

It is not difficult to show that if the signal \( f(t) \) is periodic with period \( T \), then its power \( P_f \) can be computed as follows

\[
P_f = \frac{1}{T} \int_{-T/2}^{T/2} f^2(t) dt \tag{3.38}
\]
Parseval’s relation for periodic signals says that the power content of a periodic signal is the sum of the power contents of its components in the Fourier series representation of the signal. In particular, if the exponential Fourier series coefficients of \( f(t) \) is given by the following equation:

\[
f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}
\]  

(3.39)

then, Parseval’s theorem says that the power \( P_f \) of \( f(t) \) is equal to

\[
\sum_{n=-\infty}^{\infty} |F_n|^2
\]  

(3.40)

In the above equation \( |F_n|^2 \) is the magnitude square of the complex number \( F_n \). An alternative expression of Parseval’s theorem for periodic signals, says that the power of a periodic signal \( f(t) \) is equal to

\[
a_0^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)
\]  

(3.41)

where the \( a_n \)’s and \( b_n \)’s in the above equation are the trigonometric Fourier series coefficients of the periodic signal \( f(t) \).

Besides being an alternative way of calculating the power of a periodic signal, Parseval’s relation allows us to calculate the amount of power of a periodic signal that is allocated to each one of its harmonic components.

**Example 3**

(a) Find the power contained in the first harmonic of the periodic signal \( f(t) \) of Example 1.

From the trigonometric Fourier series expansion of \( f(t) \) (see equation (3.23)) we observe that the first harmonic of \( f(t) \) is equal to

\[
\frac{4}{3\pi} \cos(2t)
\]  

(3.42)

It is not difficult to show that the amount of power contained in the first harmonic is equal to \( \frac{|\frac{4}{3\pi}|^2}{2} \) as Parseval’s relation predicts.

(b) Find the power contained in the first and second harmonics of the periodic signal \( f(t) \) of Example 1

From the trigonometric Fourier series expansion of \( f(t) \) (see equation (3.23)), we observe that the first and second harmonic of \( f(t) \) are equal to

\[
\frac{4}{3\pi} \cos(2t), \quad \frac{4}{15\pi} \cos(4t)
\]  

(3.43)

respectively.

It is not difficult to show that the amount of power contained in the first two harmonics is equal to \( \frac{|\frac{4}{3\pi}|^2 + |\frac{4}{15\pi}|^2}{2} \) as Parseval’s relation predicts.

(c) Find the power contained in the DC component of the periodic signal \( f(t) \) of Example 1.

From the trigonometric Fourier series expansion of \( f(t) \), (see equation (3.23)) we observe that the DC component of \( f(t) \) is equal to \( \frac{4}{\pi} \). Hence, the amount of power contained in the DC component is equal to \( \left(\frac{2}{\pi}\right)^2 \).
3.4 Pre-lab Questions:

1. Find both the trigonometric and the exponential Fourier series expansion of the signal \( f(t) \) in Figure 3.9. Plot amplitude versus frequency for the signal. Identify the first three harmonics and their amplitudes. Find the power content from the time domain representation of the signal and from the first three harmonics of the Fourier series expansion, and make appropriate remarks.

2. What is the difference between an energy signal and a power signal? Give two examples of each.

3.5 Implementation

In this lab you will be introduced to the relationship between pulse shapes and their spectra. There are some important concepts to recognize when performing the experiment, such as:

1. When a pulse becomes narrower in the time domain its energy becomes more spread out in frequency.
2. The envelope of the spectrum of a periodic signal is the Fourier transform of the pulse shape.

To illustrate the first concept given above, consider the Fourier Transform of a rectangular pulse shown in Fig. 3.11. It can be found from the frequency domain graph (Fig. 3.11) that 90% of the energy is concentrated within the main lobe in frequency. As the pulse width is decreased in time, or T decreases, the width of the main lobe in the frequency domain spreads, thus spreading out the energy of the signal.

To illustrate the second concept, recall that when two signals are convolved in the time domain, their transforms are multiplied in the frequency domain.

Any periodic signal can be represented mathematically as the basic pulse shape convolved with a time sequence of pulses. Thus, the spectrum of the periodic signal can be derived as shown in Figure 2.12.

3.6 Procedure:

3.6.1 Sine Wave

In this section you will observe when the signal frequency of a sine wave is shifted, the spectral line will also shift in frequency.

Time Domain

a. Set the Function Generator for a 1 KHz sine wave output. Set the amplitude to 0.5 volts peak with zero DC offset.
b. Connect the Channel 1 output of the Function Generator to Channel 1 input of the Oscilloscope using a BNC to BNC cable. Turn on the Channel 1 output of the Function Generator.
c. View the time domain waveform on the Oscilloscope.

Hint for RS RTM 3004 Oscilloscope: See EXPERIMENT 1, Section 5.1.c

Frequency Domain

a. View the frequency domain waveform of the input signal on the Oscilloscope.

Set the timebase to 2s. Wait for the sampling process shown on the Pre touch button to reach Post 100% until both time domain and frequency domain are displayed for the signal. Adjust the center frequency to 1KHz (i.e try to adjust the stop frequency to 2KHz)

Hint for RS RTM 3004 Oscilloscope: See EXPERIMENT 1, Section 5.2.a

b. Determine the amplitude and frequency displayed for the spectrum of the sine wave.
Use the measure button **Meas** to measure the amplitude and frequency displayed. When the right menu on the screen appears choose Type by clicking on Vertical button then Amplitude. Similarly by clicking on Horizontal button then Frequency. On the lower left corner the measurement value will appear.

c. Save the Screen Image to your USB Flash Drive to include in your lab report.

d. Change the Function Generator frequency to 5 KHz. Determine the amplitude and frequency displayed. (i.e don’t forget to adjust the right sampling rate and frequency ranges as in Time domain part)

e. Save the Screen Image to your USB Flash Drive to include in your lab report.

**Sampling Windows**

A sine-wave signal is initially measured as an output signal from the signal generator.

Set the following on the function generator:

**FUNCTION** Sinusoid

**FREQUENCY** 20 kHz

**AMPLITUDE** 500 mV (Vpp)

- Set the timebase for the oscilloscope to 2ms and select a maximum signal segment for the FFT.
- Select the minimum frequency band for these settings and verify that the spectrum of the sinusoidal signal is displayed correctly.
- Press the FFT button to open the menu again.
- Display the result in dBV.

The RTC3004 offers four different window functions.

a. Explain why the window functions are needed, and explain the differences between the various functions. (For detailed information please see the RTC3004 manual)

b. Compare how each of the window functions affects the spectrum

### 3.6.2 Square and Rectangular Waves

A rectangular wave contains many sine wave harmonic frequencies. When the reciprocal of the duty cycle (either the positive or negative duty cycle) is a whole number, the harmonics corresponding to multiples of that whole number will be missing. For example, if the duty cycle is 50%, then 1/0.5 = 2. Thus, the 2nd, 4th, 6th, etc. harmonics will be missing, i.e. zero. A rectangular wave with a 50% duty cycle is a square wave. For the square wave, the magnitude of the harmonic will be inversely proportional to the harmonic’s number. For example, if the magnitude of the 1st harmonic is A, then the magnitude of the 3rd harmonic is A/3, the 5th harmonic’s magnitude is A/5, etc.

a. Set the Function Generator for a 1 KHz square wave output.

b. Determine the amplitude and frequency displayed for the 1st, 3rd, 5th, and 7th harmonics in the spectrum of the square wave. Identify the zeros of the spectrum envelope (i.e. the frequency locations at which harmonic amplitudes are zero). For this question you should use the cursors. Press the **Cursor** button. Then choose the Source to the Spectrum of the required channel. Choose Type to Vertical. A cursor
will appear move it over the required harmonic in order to measure its amplitude and frequency. You can achieve that by either moving the cursor manually or clicking on the buttons **Next Peak**, **Prev. Peak** to move the cursor to different harmonics. On the lower right corner the cursor measurement will appear. Be sure to adjust the measurement type through using the right type of cursor being Horizontal or Vertical.

c. Save the Screen Image to your USB Flash Drive to include in your lab report.

d. The duty cycle of a pulse is measured as illustrated in Fig. 3.10. Switch the Function Generator to Pulse and change the duty cycle of the 1 KHz rectangular wave to 10%

e. Save the Screen Image to your USB Flash Drive to include in your lab report.

f. Change the duty cycle of the 1 KHz rectangular wave to 25%. Determine the amplitude and frequency of the 1st, 3rd, 5th, 6th, and 7th harmonics. Determine the location of zeros.

g. Save the Screen Image to your USB Flash Drive to include in your lab report.

h. Change the duty cycle of the 1 KHz square wave to 90%. Determine the amplitude and frequency of the 1st, 2nd, 3rd, 4th, and 5th harmonics. Determine the location of zeros.

i. Save the Screen Image to your USB Flash Drive to include in your lab report.

![Duty cycle of a square wave signal](image)

3.6.3 Triangular Wave

The triangular wave, like the square wave, only contains odd numbered harmonics. The magnitude of the harmonics is inversely proportional to the square of the harmonic's number. For example, if the 1st harmonic's magnitude is A, then the 3rd harmonic's magnitude will be $A/9$, the 5th harmonic's magnitude will be $A/25$, etc.

a. Set the Function Generator for a 1 KHz triangular wave (Ramp) output. Determine the amplitude and frequency of the 1st, 3rd, and 5th harmonics.

b. Determine the location of zeros.

c. Save the Screen Image to your USB Flash Drive to include in your lab report.

3.7 Calculations and Questions:

a. Compare the waveforms you obtained for rectangular waves on the basis of harmonic location, zero location, and peak spectrum amplitude (i.e. first harmonic level). Explain the results on the basis of the pulse width and signal frequency.

b. Use your data to show that the spectrum of a triangular wave is equal to the spectrum of a square wave squared.

c. The spectrum of the square wave should decay proportional to $\frac{1}{\omega}$ (where $\omega = 2\pi f$). The spectrum of the triangular wave should decay proportional to $\frac{1}{\omega^2}$. Verify this using your measurements.
Fig. 3.11: Rectangular function and its frequency representation (Fourier transform)
Fig. 3.12: Mathematical representation of a periodic signal in the time and frequency domains
3.8 Hints & Comments

Students need to distinguish between two ways of calculating the power.

Trigonometric series:

- For the DC level, they need to use the coefficient squared of the amplitude.
- For other harmonics, they should use half the coefficient squared of the amplitude.

Exponential series:

- For both DC and other harmonics, they should use coefficient squared of the amplitude.

When finding the spectrum on the oscilloscope (amplitude vs frequency), the amplitudes should always be positive because these are the RMS values of the coefficients of the exponential series harmonics.
4 EXPERIMENT: Low Pass Filter

4.1 Objective

To observe some applications of low-pass filters and to become more familiar with working in the frequency domain.

4.2 Equipment:

The equipment used in this experiment are:

- Oscilloscope: Rohde & Schwarz RTM 3004
- Function Generator, Tektronix AFG 3022B
- Digital Multimeter, Tektronix DMM 4050
- Triple Output Power Supply, Agilent E3630A
- Op Amp module TL084
- Bring a USB Flash Drive to store your waveforms.

4.3 Theory

4.3.1 Systems

A communication system consists of three major components: the transmitter, the channel and the receiver. The transmitter and the receiver are comprised of a cascade of black boxes that accept input signals, produce output signals, and they are referred to as systems. This section is devoted to understanding what a system does, and clarifying the various types of systems used in the transmitter and receiver for a communication system.

Definition: A system is a rule for producing an output signal $g(t)$ due to an input signal $f(t)$.

If we denote the rule as $T[.]$, then

$$g(t) = T[f(t)] \quad (4.1)$$

An electric circuit with some voltage source as the input and some current branch as the output is an example of a system.

Note that for two systems in cascade, the output of the first system forms the input to the second system, thus forming a new overall system. If the rule of the first system is $T_1$ and the rule of the second system is $T_2$, then the output of the overall system due to an input $f(t)$ applied to the first system is equal to $g(t)$, such that

$$g(t) = T_2[T_1[f(t)]] \quad (4.2)$$

There are a variety of classifications of systems that owe their name to their properties. In this section we will focus only in the classification of systems into the linear versus nonlinear categories and the time-invariant versus time-variant categories.
If a system is **linear** then the principle of superposition applies. The output of a system with rule that satisfies the principle of superposition exhibits the following property:

\[ T[a_1f_1(t) + a_2f_2(t)] = a_1T[f_1(t)] + a_2T[f_2(t)] \] (4.3)

Where \( a_1 \) and \( a_2 \) are arbitrary constants. A system is **nonlinear** if it is not linear. A linear system (friendly system) is usually described by a linear differential equation of the following form:

\[ a_n(t)g^n(t) + a_{n-1}g^{n-1}(t) + \ldots + a_1(t)g^1(t) + a_0(t)g^0(t) = f(t) \] (4.4)

Where \( g(t) \) designates the output of the system, while \( f(t) \) designates the input of the system. In the above equation \( g^k(t) \) denotes the \( k \)-th time derivative of the function \( g(t) \).

Virtually every system that you considered in your circuit classes (e.g. first order R - C , R - L circuit, second order R - L - C circuits) is an example of linear systems. Any circuit that has components, whose \( v \)-\( i \) curve is nonlinear (e.g., a diode) is likely to be a nonlinear system.

Another useful classification of systems is into the categories of Time-Invariant versus Time-Varying systems. A system is **time-invariant** if a time-shift in the input results in a corresponding time-shift in the output. Quantitatively, a system is time-invariant if

\[ g(t - t_0) = T[f(t - t_0)] \] (4.5)

For any \( t_0 \) and any pair of \( f(t) \) and \( g(t) \), where \( f(t) \) designates an input to the system and \( g(t) \), denotes its corresponding output. A system is **time-varying** if it is not time-invariant.

As we emphasized before, a linear system is described by a linear differential equation of the form shown in equation (4.4). If in the differential equation (4.4) the coefficients \( a_k(0 \leq k \leq n) \) are constants and not functions of time, then we are dealing with a linear time-invariant system. Otherwise, we are dealing with a linear time-varying system. Examples of linear time-varying systems from your circuit classes were circuits consisting of R, L and C components and involving one or more switches that were ON or OFF at special instances.

### 4.3.2 The Convolution Integral

The convolution integral appears quite often in situations where we deal with a linear time invariant system (LTI). If this system is excited by an input \( f(t) \) and the impulse response of the system is \( h(t) \), then the output \( g(t) \) of the system is equal to the convolution of \( f(t) \) and \( h(t) \), as illustrated below

\[ g(t) = f(t) * h(t) = \int_{-\infty}^{\infty} f(t - \tau)h(\tau)d\tau = \int_{-\infty}^{\infty} h(t - \tau)f(\tau)d\tau \] (4.6)

Note that the impulse response of a linear, time-invariant system is defined to be the output of the system due to an input that is equal to an impulse function, located at \( t = 0 \). In this manual we are going to demonstrate the convolution of two rectangular pulses.

### 4.3.3 Convolution Examples

Consider two rectangular pulses, \( f_1 \) and \( f_2 \), depicted in Fig. 4.1. We are going to find the convolution of these two rectangular pulses. We designate the convolution by \( f_3(t) \). That is
We want to compute the convolution of \( f_1(t) \) and \( f_2(t) \). It seems that we have to compute the convolution integral for infinitely many instances of time \( t \) (see equation (4.7)). A more careful observation though allows us to distinguish the distinct \( t \)-range over which the convolution integral needs to be evaluated. One way of finding the distinct \( t \)-range is by remembering that equation (4.7) tells us that we need to slide the rectangle \( f_1 \) with respect to the stationary rectangle \( f_2 \) and identify the product of these two pulses over the common interval of overlap. It is not difficult to show that the sliding rectangle \( f_1 \) (actually \( f_1(t - \tau) \)) is nonzero over the \( \tau \)-interval \( (t - 2, t - 1) \). We now distinguish five cases:

Case A: \( f_1(t - \tau) \) is to the left of \( f_2(\tau) \), and \( f_1 \) and \( f_2 \) do not overlap (see Fig. 4.2).

In order for Case A to be valid we have to satisfy the constraint that \( t - 1 < 3 \) or \( t < 4 \). Then it is easy to show that

\[
f_3(t) = 0 \tag{4.8}
\]

Case B: \( f_1(t - \tau) \) is to the left of \( f_2(\tau) \), and overlap partially from the left (see Fig. 4.3)

In order for Case B to be valid we have to satisfy the constraint that \( 3 < t - 1 < 4 \) or \( 4 < t < 5 \). Then it is easy to show that

\[
f_3(t) = \int_3^{t-1} d\tau = t - 4 \tag{4.9}
\]

Case C: \( f_1(t - \tau) \) is completely overlapping with \( f_2(\tau) \) (see Fig. 4.4).

In order for Case C to be valid we have to satisfy the constraint that \( 4 < t - 1 < 5 \) or \( 5 < t < 6 \). Then it is easy to show that

\[
f_3(t) = \int_{t-2}^{t-1} d\tau = (t - 1) - (t - 2) = 1 \tag{4.10}
\]
Case D: $f_1(t - \tau)$ is to the right of $f_2(\tau)$, and $f_1$ overlap $f_2$ partially from the right (see Fig. 4.5).

In order for Case D to be valid we have to satisfy the constraint that $5 < t - 1 < 6$ or $6 < t < 7$. Then it is easy to show that

$$f_3(t) = \int_{t-2}^{5} d\tau = 5 - (t - 2) = 7 - t$$  \hspace{1cm} (4.11)

Case E: $f_1(t - \tau)$ is to the right of $f_2(\tau)$, and $f_1$ and $f_2$ do not overlap (see Fig. 4.6).

In order for Case E to be valid we have to satisfy the constraint that $6 < t - 1$ or $t > 7$. Then is easy to show that

$$f_3(t) = 0$$  \hspace{1cm} (4.12)

Hence, combining all the previous cases, we get

$$f_3(t) = \begin{cases} 
0 & \text{for } t < 4 \\
(t - 4) & \text{for } 4 < t < 5 \\
1 & \text{for } 5 < t < 6 \\
7 - t & \text{for } 6 < t < 7 \\
0 & \text{for } t > 7 
\end{cases}$$  \hspace{1cm} (4.13)

A plot of the function $f_3(t)$ is shown in Fig. 4.7.

Based on our previous computations we can state certain rules pertaining to the convolution of two rectangles. These rules can be proven following the technique laid out in the previous example. These rules can be used to find the convolution of two rectangle pulses, without actually having to compute the convolution.
Fig. 4.4: Convolution under Case c

Fig. 4.5: Convolution under Case D

Fig. 4.6: Convolution under Case E

Fig. 4.7: Function $f_3(t)$ (the result of the convolution of $f_1(t)$ and $f_2(t)$)
4.3.4 Golden Rules for the Convolution of two Rectangle Pulses

Consider a rectangular pulse $f_1(t)$ with amplitude over the interval $[x_1, x_2]$, and another rectangular pulse $f_2(t)$ with amplitude over the interval $[y_1, y_2]$. Denote their convolution by $f_3(t)$. Then,

1. The function $f_3(t)$ is a trapezoid.
2. The starting point of the trapezoid is at position $x_1 + y_1$.
3. The first breakpoint of the trapezoid is
   (a) At position $x_1 + y_2$ if pulse $f_2(t)$ is of smaller width than pulse $f_1(t)$.
   (b) At position $x_2 + y_1$ if pulse $f_1(t)$ is of smaller width than pulse $f_2(t)$.
4. The second breakpoint of the trapezoid is
   (a) At position $x_2 + y_1$ if pulse $f_2(t)$ is of smaller width than pulse $f_1(t)$.
   (b) At position $x_1 + y_2$ if pulse $f_1(t)$ is of smaller width than pulse $f_2(t)$.
5. The end point of the trapezoid is at position $x_2 + y_2$.
6. The maximum amplitude of the trapezoid is
   (a) Equal to $A_1 A_2 (x_2 - x_1)$ if the $f_1(t)$ is of smaller width.
   (b) Equal to $A_1 A_2 (y_2 - y_1)$ if the $f_2(t)$ is of smaller width.

A plot of $f_1(t)$, $f_2(t)$ and $f_3(t)$ is shown in Fig. 4.8. The technique, described above, to compute the convolution of two rectangular pulses can be generalized, in a trivial way, to compute the convolution of arbitrary shaped, finite-duration pulse.

All of the above rules pertaining to the convolution of two rectangular pulses can be proven rigorously. They have been verified above for an example case. In the special case where the pulse (rectangle) $f_1(t)$ and $f_2(t)$ are defined to be nonzero over an interval of the same width convolution $f_3(t)$ turns out to be a triangle. Rules 2 and 5, stated above, for the convolution of two rectangular pulsed extend for the case of the convolution of arbitrary shaped, finite-duration pulses.

Some useful properties of the convolution are listed below. These properties can help us compute the convolution of signals that are more complicated than the rectangular pulses.
1. Commutative Law:

\[ f_1(t) * f_2(t) = f_2(t) * f_1(t) \]  
(4.14)

2. Distributive Law:

\[ f_1(t) * [f_2(t) + f_3(t)] = f_1(t) * f_2(t) + f_1(t) * f_3(t) \]  
(4.15)

3. Associative Law:

\[ f_1(t) * [f_2(t) * f_3(t)] = [f_1(t) * f_2(t)] * f_3(t) \]  
(4.16)

4. Linearity Law:

\[ [\alpha f_1(t)] * f_2(t) = \alpha [f_1(t) * f_2(t)] \]  
(4.17)

Where \( \alpha \) in the above equation is a constant.

4.3.5 Impulse Function

The impulse function showed up in the discussion of the convolution integral and linear time-invariant system. We said then, that the convolution integral allows us to compute the output of a linear time-invariant system if we know the system's impulse response. The impulse response of a system is defined to be the response of the system due to an input that is the impulse function located at \( t = 0 \).

The impulse or delta function is a mathematical model for representing physical phenomena that take place in a very small time duration, so small that it is beyond the resolution of the measuring instrument involved, and for all practical purposes, their duration can be assumed to be equal to zero. Examples of such phenomena are a hammer blow, a very narrow voltage or current pulse, etc. In the precise mathematical sense, the impulse signal, denoted by

\[ \delta(t) \]

is not a function (signal), it is a distribution or a generalized function. A distribution is defined in terms of its effect on another function (usually called "test function") under the integral sign.

The impulse distribution (or signal) can be defined by its effect on the "test function" \( \phi(t) \), which is assumed to be continuous at the origin, by the following relation:

\[
\int_a^b \phi(t) \delta(t) = \begin{cases} 
\phi(0) & \text{for } a < 0 < b \\
0 & \text{otherwise}
\end{cases}
\]

This property is called the sifting property of the impulse signal. In other words, the effect of the impulse signal on the "test function" \( \phi(t) \) under the integral sign is to extract or sift its value at the origin. As it is seen, \( \delta(t) \) is defined in terms of its action on \( \phi(t) \) and not in terms of its value for different values of \( t \).
One way to visualize the above definition of the impulse function is to think of the impulse function as a function determined via a limiting operation applied on a sequence of well-known signals. The defining sequence of signals is not unique and many sequences of signals can be used, such as

1. Rectangular Pulse:

\[
\delta(t) = \lim_{\tau \to \infty} \frac{1}{\tau} \left[ U(t + \frac{\tau}{2}) - U(t - \frac{\tau}{2}) \right]
\]

2. Triangular Pulse:

\[
\delta(t) = \lim_{\tau \to 0} \frac{1}{\tau} \left[ 1 - \frac{|t|}{\tau} \right]
\]

3. Two-sided exponential:

\[
\delta(t) = \lim_{\tau \to 0} \frac{1}{\tau} e^{-\frac{|t|}{\tau}}
\]

4. Gaussian Pulse:
The above functions are depicted in Fig. 4.9.

The impulse function that we have defined so far was positioned at time \( t = 0 \). In a similar fashion we can define the shifted version of the impulse function. Hence, the function \( \delta(t - t_0) \) designates an impulse function located at position (time) \( t_0 \), and it is defined as follows:

\[
\int_a^b \phi(t)\delta(t - t_0) = \begin{cases} 
\phi(t_0) & \text{for } a < t_0 < b \\
0 & \text{otherwise}
\end{cases}
\]

The impulse function has a number of properties that are very useful in analytical manipulations involving impulse functions. They are listed below:

- **Area (Strength):**
  The impulse function \( \delta(t) \) has unit area. That is

\[
\int_a^b \delta(t - t_0)dt = 1 \text{ where } a < t_0 < b
\]

- **Amplitude:**

\[
\delta(t - t_0) = 0 \text{ for all } t \neq t_0
\]

- **Graphic representation:**
  See Fig. 4.10.

- **Symmetry:**

\[
\delta(t) = \delta(-t)
\]
• Time Scaling:

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

• Multiplication by a time function:

$$f(t)\delta(t - t_0) = f(t_0)\delta(t - t_0)$$

• Relation to the Unit Step Function:

The unit step function is the function defined by

$$U(t - t_0) = \begin{cases} 
1 & \text{for } t \geq t_0 \\
0 & \text{for } t < t_0 
\end{cases}$$

It is not difficult to see that

$$\int_{-\infty}^{t} \delta(\tau - t_0)d\tau = U(t - t_0)$$

And that

$$\delta(t - t_0)d\tau = \frac{d}{dt}U(t - t_0)$$

Similar to the definition of $\delta(t)$ we can define $\delta^{(1)}(t)$, $\delta^{(2)}(t)$, ..., $\delta^{(n)}(t)$ the generalized derivatives of $\delta(t)$ by the following equation:

$$\int_{a}^{b} \delta^{(n)}(t)\phi(t)dt = \begin{cases} 
(-1)^n \frac{d^n\phi(t)|_{t=t_0}}{dt^n} & \text{for } a < 0 < b \\
0 & \text{otherwise} 
\end{cases}$$

We can generalize this result to

$$\int_{a}^{b} \delta^{(n)}(t - t_0)\phi(t)dt = \begin{cases} 
(-1)^n \frac{d^n\phi(t)|_{t=t_0}}{dt^n} & \text{for } a < t_0 < b \\
0 & \text{otherwise} 
\end{cases}$$

For even values of $n$, $\delta^{(n)}(t)$ is even, and for odd values of $n$, $\delta^{(n)}(t)$ is odd.
4.3.6 The Frequency Transfer Function

Consider a linear-time invariant system, which is described by the following differential equation.

\[ \sum_{m=0}^{M} a_m \frac{d^m g(t)}{dt^m} = \sum_{k=0}^{K} b_k \frac{d^k f(t)}{dt^k} \]  

(4.18)

In the above equation, \( f(t) \) represents the input to the system, and \( g(t) \) represents the output of the system. The expression \( \frac{d^m f(t)}{dt^m} \) denotes 0-th derivative of \( f(t) \); same convention holds for \( g(t) \).

Assume for a moment that the input \( f(t) \) to the above system is equal to \( e^{j\omega t} \). Then, we prove that the output of the system is equal to \( g(t) = H(\omega) e^{j\omega t} \) with

\[ H(\omega) = \frac{\sum_{k=0}^{K} b_k (j\omega)^k}{\sum_{m=0}^{M} a_m (j\omega)^m} \]  

(4.19)

In actuality we are contending that if the input to the system is a complex exponential function \( e^{j\omega t} \) then the output of the system will be the same exponential function times a constant \( H(\omega) \) which depends on the input parameters \( \omega \) and the system parameters (the \( a \)'s and \( b \)'s) when we are referring to \( H(\omega) \) as being a constant we mean that it is independent of time. The constant \( H(\omega) \) is called the transfer function of the system. When the Fourier transform is introduced it will be easy to prove that the transfer function of a system is the Fourier transform of the impulse response of the system. We have already stated that the impulse response of a linear time invariant system describes the system completely. We can make a similar statement about the transfer function of a linear, time-invariant system. That is, if we know the transfer function of a LTI system we can compute the output of this system due to an arbitrary input. Since, we are still operating in the context of Fourier series expansions, and since Fourier series expansions are most appropriate for periodic signals, consider for a moment a periodic input \( f(t) \) applied to the system of equation (4.18). Obviously, \( f(t) \) has a Fourier series expansion whose form is given below.

\[ f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \]  

(4.20)

It is not difficult to demonstrate that in this case the output \( g(t) \) of our system will be equal to

\[ g(t) = \sum_{n=-\infty}^{\infty} H(n\omega_0) F_n e^{jn\omega_0 t} \]  

(4.21)

Equation (4.21) validates our claim that the frequency transfer function of a linear time invariant system is sufficient to describe the output of the system due to an arbitrary input (at least for the case of an input which is periodic). After the Fourier transform is introduced we will be able to extend this result to aperiodic inputs as well. Looking at equation (4.21) more carefully we can make a number of observations:

Observation 1: If the input to a linear time-invariant system is periodic with period \( T_0 = \frac{2\pi}{\omega_0} \) then the output of this system is periodic with the same period.

Observation 2: Only the frequency components that are present in the input of a LTI system can be present in the output of the LTI system. This means that a LTI system cannot introduce new frequency components other than those present in the input. In other words all systems capable of introducing new frequency are either nonlinear systems or time-varying systems or both.
4.3.7 Frequency Transfer Function Example

Let \( f(t) \) be a signal as shown in Fig. 4.11. This signal is passed through a system (filter) with transfer function as shown in Figure 4.12. Determine the output \( g(t) \) of the filter.

We first start with the Fourier series expansion of the input. This can be obtained by following Fourier series formulas.

\[
f(t) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \cos[2\pi(2n+1)10^5t] = \frac{2}{\pi} e^{j2\pi10^5t} + \frac{2}{3\pi} e^{-j6\pi10^5t} - \frac{2}{3\pi} e^{j6\pi10^5t} + \frac{2}{5\pi} e^{j10\pi10^5t} - \frac{2}{5\pi} e^{-j10\pi10^5t} + \ldots
\]  

To find the output of the system we need to find the output of the system due to every complex exponential involved in the expansion of \( f(t) \). A closer inspection of \( H(\omega) \) though tells us that only the complex exponential within the passband of the filter will actually produce an output (the passband is the range of frequencies from \(-12\pi10^5\) rads to \(12\pi10^5\) rads.) Hence, using the already derived formula that gives the output of an LTI system due to a periodic input, we can write:
4.3.8 Filters

Filtering has a commonly accepted meaning of separation - something is retained and something is rejected. In electrical engineering, we filter signals usually frequencies. A signal may contain single or multiple frequencies. We reject frequency components of a signal by designing a filter that provides attenuation over a specific band of frequencies, and we retain components of a signal through the absence of attenuation or perhaps even gain over a specified band of frequencies. Gain may be defined as how much the input is amplified at the output.

Filters are classified according to the function they have to perform. Over the frequency range we define pass band and stop band. Ideally pass band is defined as the range of frequencies where gain is 1 and attenuation is 0, and stop band is defined, as the range of frequencies where the gain is 0 and the attenuation is infinite.

Filters can be mainly classified as low pass, high pass, band pass and band stop filters.

A low pass filter can be characterized by the property that the pass band extends from frequency \( \omega = 0 \) to \( \omega = \pm \omega_c \) where \( \omega_c \) is known to be the cutoff frequency. (See Fig. 4.13a)

A high pass filter is the complement of a low pass filter. Here the previous frequency range \( \omega = 0 \) to \( \omega = \pm \omega_c \) is the stop band and the frequency range from \( \omega = \omega_c \) to positive infinity and from \( \omega = -\omega_c \) to negative infinity is the pass band. (See Fig. 4.13b)

A band pass filter is defined as the one where frequencies from \( \omega_1 \) to \( \omega_2 \) are passed while all other frequencies are stopped. (See Fig. 4.13c)

A band stop filter is the complement of the band pass filter. Frequencies from \( \omega_1 \) to \( \omega_2 \) are stopped here and all other frequencies are passed. (See Fig. 4.13d)

In Fig. 4.14 below a practical (non-ideal) 6th order Butterworth low-pass filter is shown.

In this the various filter characteristics such as pass band, cut-off frequency are clearly indicated. In Fig. 4.15 we are illustrating how the filter characteristics change with the order of the filter.

The following Fig. 4.15 shows how the order of a filter changes the frequency response. The higher the order, the closer the filter behaves as ideal.

Filters are key components in communication systems. An ideal filter will only allow a specified set of frequencies to pass from input to output with equal gain. However, no filter is ideal, and as a result there are many types of filters, that are used in modeling an ideal filter closely in one or more aspects. Filters have many applications besides separating a signal form surrounding noise or other signals. Some of these applications are listed below.

1. Integration of a signal
2. Differentiation of a signal
3. Pulse shaping
4. Correcting for spectral distortion of a signal
5. Sample and hold circuits

Two Butterworth low pass filters (LPF’s) will be used in this lab: a first order and a fourth-order Butterworth filter. The general transfer function of a Butterworth LPF is:

\[
g(t) = \frac{2}{\pi} e^{j(2\pi 10^5 t + \frac{\pi}{2})} + \frac{2}{\pi} e^{-j(2\pi 10^5 t + \frac{\pi}{2})} - \frac{2}{\pi} e^{j(6\pi 10^5 t + \frac{\pi}{2})} - \frac{2}{\pi} e^{-j(6\pi 10^5 t + \frac{\pi}{2})} + \frac{2}{\pi} e^{j(10\pi 10^5 t + \frac{\pi}{2})} + \frac{2}{\pi} e^{-j(10\pi 10^5 t + \frac{\pi}{2})}
\]
Fig. 4.13: The magnitude of the frequency transfer function of low-pass, high-pass, band-pass, band-stop and all-pass. In all cases the responses are ideal.

Fig. 4.14: $6^{th}$ order Low-pass Butterworth Filter
Fig. 4.15: Butterworth Filters (How the frequency response changes with order)

\[ |H(\omega)|^2 = \frac{A}{(1 + \omega/\omega_0)^{2n}} \]  

(4.24)

Where \( \omega_0 \) = the filter cutoff frequency and 
\( n \) = the order of the filter

The actual transfer functions of the filters that are going to be used in this lab are:

1st order LPF:

\[ |H(s)| = \frac{1}{(1 + RCs)} \]  

(4.25)

4th order LPF:

\[ |H(s)| = \frac{(1 + \frac{R_2}{R_1})(1 + \frac{R_3}{R_4})}{((RC)^4s^4 + (3 - (\frac{R_1+R_2}{R_1})RCs + 1)(RC)^2s^2 + (3 - (\frac{R_1+R_2}{R_1})RCs + 1))} \]  

(4.26)

where \( \omega_0 = \frac{1}{RC} \) for both filters.

Two applications for low pass filters will be illustrated in this lab:

1. A LPF used as an integrator
2. A LPF used as a first harmonic isolator (square-to-sine conversion).

**Integrator**

The Laplace representation of integration is \( \frac{1}{s} \) (where \( s = j\omega \)). A first order low pass filter can approximate an integrator when \( \omega >> \omega_0 \) \( \left( \frac{V_{out}}{V_{in}} = \frac{1}{s\omega_0} \right) \)

For an \( n^{th} \) order low-pass filter, the transfer function will approximate \( n \) integrators in series at \( \omega >> \omega_0 \) \( \left( \frac{V_{out}}{V_{in}} = \frac{1}{s\omega_0^n} \right) \)

For simplicity, in this lab a first order LPF will be used to perform a single integration on a signal.

**Square-to-sine-Converter**

Any periodic signal can be expressed as an infinite sum of orthogonal sinusoids. For orthogonality, each sinusoid (or “harmonic”) must have a different frequency, and frequency must be an integer multiple of the frequency of the periodic signal.
For example, if the frequency of a square wave is $f_s$, then the $n^{th}$ harmonic will have a frequency of $nf_s$ (where $n$ is any integer from one to infinity). You will see that some of these harmonics will have zero or negative amplitude.

The harmonics of a square wave of frequency $f_s$ are shown in Fig. 4.16 (Not to scale); the DC component of the square wave is assumed to be equal to zero.

If the square wave excites the input of the low pass filter such that the first harmonic is in the pass band of the filter transfer function and the remaining harmonics are outside of the pass band, then the output of the low pass filter will be a single-tone signal (sinusoid).

### 4.4 Simulation

We will use MATLAB simulation to see the response of our filter circuits. For simulation you will need the transfer functions of both filter circuits shown in Fig. 4.20 and Fig. 4.21. The transfer functions were given earlier in this manual (See equation 4.25 and 4.26.)

In both filters and $R = 1.6k\Omega$ and $C = 0.01F$. For the second filter $R_1 = 10k\Omega$, $R_2 = 12k\Omega$, $R_3 = 10k\Omega$, $R_4 = 1.5k\Omega$ and $C = 0.01F$. With the above values for resistors and capacitors the transfer functions will have the following form (notice that the coefficients are in descending order of $s$).

$$H_1 = \frac{1}{1 + (RC)s}$$

$$H_2 = \frac{2.2 \times 1.5}{(RC)^4s^4 + 2.65(RC)^3s^3 + 3.48(RC)^2s^2 + 2.65RCs + 1}$$

Construct a frequency array as following.

```matlab
w=[0:100:100000];
```

Construct the coefficient vectors as follows:

```matlab
b=[coefficients of numerator separated by comma];
```

```matlab
a=[coefficients of denominator separated by comma];
```

Use the MATLAB 'freqs' function to generate frequency response of the corresponding filter.

Note: The freqs function is not available in all MatLab installations. It is installed in all EE Laboratories.

Plot the frequency response of both filters in a single graph using the following a MATLAB function.

Apply the above procedure for both filters. Note that the frequency response you obtained from the MATLAB simulation that is gain versus angular frequency in radian. You can also plot gain versus frequency (Hz). In that case you have to use the $f = \frac{\omega}{2\pi}$ vector as the parameter in plot function.
The simulated gain versus frequency (Hz) plot for both filters (2nd order and 6th order) is shown above as Fig. 4.17.

Compare the response of two filters. After the hardware experiment compare the experimental results with the simulated results.

### 4.5 Prelab Questions:

1. Calculate the Trigonometric Fourier Series of a square wave with the parameters shown in Fig. 4.18 (assume it is periodic with periodic $T$):

You may find it easier to add a DC level of $A$ volts. This will only change your spectrum at $f = 0$, where you can subtract the DC back out.

2. Assuming an ideal filter with a cutoff frequency of $\omega_c$ ($\omega_c = \frac{2\pi}{T}$), show graphically, in the frequency domain, why the first harmonic of the square wave should be the only harmonic at the output.

3. Suppose you have an input signal with frequency components from 0 to 2 KHz. Is it possible to design a filter, which will produce an output with frequency components 4 to 6 KHz? Why or why not?

4. What is the difference between dB/octave and dB/decade?

### 4.6 Procedure

#### 4.6.1 Components

In this lab, we will use Op Amp module TL084, which is shown in Fig. 4.19.
4.6.2 Filters

a) Build the first order low-pass filter shown in Fig. 4.20.

b) Build the fourth order low-pass filter shown in Fig. 4.21. Keep both circuits built for the remainder of the lab.

4.6.3 Calculating Filter Parameters

Cutoff Frequency

a) From the function generator, obtain a 0.2 volt, low frequency (10 or 100 Hz range) sine wave. It is important to use a sine wave because there is only one harmonic in a sinusoid, and you only want to test the transfer function at one point in frequency at time. Connect the function generator output to the filter input.

b) Connect the filter output to either the oscilloscope or to the DMM4050 Digital Multimeter. When using the DMM, select dB measurements.

c) Vary the input frequency by a factor of 2 or 3 either way and make sure the output level doesn’t vary significantly. If it doesn’t, then your input frequency is within the pass band. When you know you are within the pass band, measure the output voltage level (or dB level). This will be your reference level. (i.e. Use...
d) Vary the input frequency until the output level drops 3 dB below the reference level. You can achieve that by keeping the measure on Vpp and rotating the knob on the functional generator while the cursor is on the frequency measurement. On the oscilloscope, -3 dB is equivalent to the voltage level reduced by a factor of .707 (or, at -3dB point, $V_{out}(-3dB) = .707(V_{out(ref)})$). Note this frequency. This is your cutoff frequency.

N.B. Be sure to fit the two scales manually by knobs in case Autoset doesn’t set well.

Op-amp voltage +15V, it can reach +20V if clipping voltage with Autoset then manually change timebase.

**Passband Gain**

a) Using the function generator, generate a 0.2 volt sine wave at a frequency that you know to be within the filter passband. Connect the function generator output to the filter input.

b) Measure the output voltage (0V). The filter passband gain is equal to $\frac{V_{out}}{V_{in}} = \frac{V_{out}}{0.2}$

c) Repeat steps 1 and 2 with a frequency that is a factor of 2 higher than the first frequency used. If you take the difference between the two calculated values of attenuation, in dB, this will be your rate of rolloff in dB/octave. If you want to convert, $(x)$ dB/octave corresponds to $(x \times (20/6))$ dB/decade.

**4.6.4 Square Wave Spectrum**

a) Set the function generator for a square wave of 0.2 volts amplitude and 1 kHz frequency. Connect the output of the function generator to Channel 1 input of the oscilloscope.

b) Use the Oscilloscope to display the spectrum analysis. You should see the first six nonzero harmonics of the square wave displayed. Record the amplitude (in dBV) of the first six nonzero harmonics and the...
frequency of each harmonic. Be sure to adjust the timebase to 100 ms or above, for a display that can show you the needed spectrum. Choose measure frequency by pressing the Meas button and changing the Type, choose Cursor by pressing its button. A menu will appear, on Source choose FFT: Spectrum. Then press on Next Peak to measure the frequency at that harmonic.

4.6.5 Integrator

a) Generate a square wave of amplitude 5V (peak) and frequency = 10 \times \omega_c \ (ten \ times \ the \ cutoff \ frequency). Apply this signal to the input of the filter from Fig. 4.20 (the first order filter)

b) Sketch the output of the filter from the oscilloscope, recording signal peak-to-peak amplitude. You can achieve that by pressing the Meas button and adjusting the Type to Peak Peak by plugging the Basic menu button. The peak to peak amplitude measurement will appear on the bottom left corner of the oscilloscope screen.

4.6.6 Square-to-sine Conversion (First Harmonic Isolation)

a) Obtain a 0.2 volt peak square wave with frequency f = 0.75 \times f_c \ from \ the \ function \ generator.

b) Apply the signal from (a) to the input of the filter in Fig. 4.20 Sketch the output signal in both the time and frequency domains, scaling axes.

c) Repeat b) with the filter in Fig. 4.21.

4.7 Calculations and Questions

1. Convert the values you measured for the square wave in Section 4.5.4 in dBV to volts and compare to your Fourier Series Coefficients calculated in the Prelab. Recall that the measured amplitudes are RMS amplitudes, so the values must be multiplied by the square root of 2.

2. Compare the expected results for filter gain and filter roll off rate to your measured results.

3. Using the waveform shown in question (1), Integrate (in time) to show that the integral of a square wave is a triangular wave. What peak amplitude do you calculate? Recall that you have to solve an indefinite integral to get an answer in functional form.

4. When the first harmonic of the input signal is well beyond the filter cutoff frequency, a first order low pass filter cutoff frequency, a first order low pass filter approximates an integrator (shown in ‘Integrator’ section of lab introduction). Show that this transfer function, when applied to your square wave, will closely predict the output amplitude you measured. Note that this transfer function can be reduced to a constant multiplied by the integral of the input signal. Also note that your DC level may vary from that calculated, because the DC level will depend on at what point on the square wave the integration began.
4.8 Hints & Comments

- Students need to answer all the questions for the two types of filters (First Order Low Pass Filter and Fourth Order Low Pass Filter).

- When implementing the Fourth Order Low Pass Filter, students should use the Op-Amp module TL084. This module consists of four Op-Amps. The module needs to be stimulated with a DC voltage. A typical DC voltage value is $+15\,V$ to $+V_{CC}$ Pin 4 and $-15\,V$ to $-V_{EE}$ Pin 11 from the DC Power Supply.

- When computing the Roll Off value of the filter, the frequencies used should be double the cut off frequency of the filter or more.

- When using the Low Pass Filters at a high frequency, above the cut off frequency (110 kHz) with an input signal of a Square wave, the output should be a triangular wave.

- When using a higher order Low Pass Filter, the output result should be smoother (better resolution).
5 EXPERIMENT: Amplitude Modulation

5.1 Objective

To understand the amplitude modulation.

5.2 Equipment:

The equipment used in this experiment are:

- Oscilloscope: Rohde & Schwarz RTM 3004
- Function Generator: Tektronix AFG 3022B
- Digital Multimeter, Tektronix DMM 4050
- Triple Output Power Supply, Agilent E3630A
- Op Amp module TL084
- Bring a USB Flash Drive to store your waveforms.

5.3 Theory

5.3.1 From Fourier series to the Fourier Integral

The Fourier series is a means for expanding a periodic signal in terms of complex exponentials. This expansion considerably decreases the complexity of the description of the signal, and simultaneously, this expansion is particularly useful when we analyze LTI systems.

We can extend the idea of the Fourier series representation from periodic signals to the case of nonperiodic signals. That is, the expansion of a nonperiodic signal in terms of complex exponentials is still possible. However, the resulting spectrum is not discrete any more. In other words, the spectrum of nonperiodic signals covers a continuous range of frequencies. To be able to demonstrate our point, consider for a moment the periodic signal shown in Fig. 5.1.

Fig. 5.1: A periodic signal $f_T(t)$

We call this periodic signal $f_T(t)$ to remind ourselves that we are dealing with a periodic signal of period $T$. We can write:

$$f(t) = \begin{cases} f_T(t) & \frac{-T}{2} < t < \frac{T}{2} \\ 0 & otherwise \end{cases} \quad (5.1)$$

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where \( f(t) \) is equal to the periodic signal \( f_T(t) \) over the interval \( (-\frac{T}{2}, \frac{T}{2}) \) and zero otherwise. Obviously, \( f_T(t) \) has a Fourier series expansion, such that

\[
f_T(t) = \sum_{n=-\infty}^{\infty} F_n e^{j n \omega_0 t} \tag{5.2}
\]

With

\[
F_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f_T(t) e^{-j n \omega_0 t} \, dt
\]

\[
= \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-j n \omega_0 t} \, dt \tag{5.3}
\]

Also, since \( f_T(t) \) and \( f(t) \) are equal over the interval \( (-\frac{T}{2}, \frac{T}{2}) \) we can write that

\[
f(t) = \sum_{n=-\infty}^{\infty} F_n e^{j n \omega_0 t}
\]

\[
= \sum_{n=-\infty}^{\infty} \frac{1}{T} F(\omega_n) e^{j \omega_n t} \tag{5.4}
\]

where

\[
F(\omega_n) = TF_n \tag{5.5}
\]

and

\[
\omega_n = n \omega_0 \tag{5.6}
\]

Let us now define

\[
\Delta \omega = \omega_{n+1} - \omega_n = \omega_0 = \frac{2\pi}{T} \tag{5.7}
\]

Then, from equation (5.7) we get

\[
f(t) = \sum_{n=-\infty}^{\infty} \frac{\Delta \omega}{2\pi} F(\omega_n) e^{j \omega_n t} \tag{5.8}
\]

Let now \( T \to \infty \), or equivalently, \( \Delta \omega \to 0 \). It is easy to see that the equation (5.8) becomes:

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j \omega t} \, d\omega \tag{5.9}
\]
Also, it is not difficult to show that

\[
F(\omega) = \int_{-\infty}^{\infty} f(t) e^{j\omega t} \, dt \tag{5.10}
\]

Equations (5.9) and (5.10) are referred to as the Fourier Transform pair, that allows us to transform the aperiodic time signal \( f(t) \) into the frequency signal \( F(\omega) \), and vice versa. Based on the aforementioned discussion we can now formally introduce the Fourier transform theorem.

### 5.3.2 Fourier Transform Theorem

If the signal \( f(t) \), satisfies certain conditions known as the Dirichlet conditions, namely,

1. \( f(t) \) is absolutely integrable on the real line, that is,
   \[
   \int_{-\infty}^{\infty} |f(t)| \, dt < \infty \tag{5.11}
   \]

2. The number of maxima and minima of \( f(t) \) in any finite interval of the real line is finite.
3. The number of discontinuities of \( f(t) \) in any finite interval of the real line is finite.

Then, the Fourier transform (FT) of \( f(t) \), defined by

\[
F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt \tag{5.12}
\]

exists and the original signal can be obtained from its FT by following the rule

\[
f_\pm = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega \tag{5.13}
\]

where

\[
f_\pm(t) = \begin{cases} 
    f(t), & \text{if } f(t) \text{ is continuous at } t \\
    \frac{f(t^+) + f(t^-)}{2}, & \text{if } f(t) \text{ is not continuous at } t
\end{cases} \tag{5.14}
\]

Observation 1: The Fourier transform, \( F(\omega) \) of a signal \( f(t) \), is, in general, a complex function. Its magnitude \( |F(\omega)| \) and phase \( \angle F(\omega) \) represent the amplitude and phase of various frequency components of \( f(t) \).

Observation 2: If the independent variable in the FT of a signal \( f(t) \) is chosen to be the frequency \( f \) instead of the angular frequency \( \omega \), we have the following Fourier transform pair:

\[
X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} \, dt < \infty \tag{5.15}
\]

and

\[
x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} \, df \tag{5.16}
\]
5.3.3 Fourier Series to Fourier Transform Example

Let \( f_{T_0}(t) \) be a periodic signal depicted in Fig. 5.2.

The Fourier series expansion of this signal, of period \( T_0 \), is given by the following equation.

\[
 f_{T_0}(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t} \tag{5.17}
\]

where

\[
 F_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} f(t) e^{-jn\omega_0 t} dt \tag{5.18}
\]

and

\[
 \omega_0 = \frac{2\pi}{T_0} \tag{5.19}
\]

We can compute \( F_n \)'s as follows:

\[
 F_n = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} f(t) e^{-jn\omega_0 t} dt \\
 = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} f(t) e^{-j\frac{2\pi n}{T_0} \tau} dt \\
 = \frac{1}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} f(t) e^{-j\frac{2\pi n}{\tau} \tau} dt \\
 = \frac{1}{T_0} \left( \frac{T_0}{jn2\pi} \right) \left[ e^{-j\frac{n\pi}{\tau_0}} - e^{j\frac{n\pi}{\tau_0}} \right] \\
 = \frac{\sin(\frac{n\pi}{\tau_0}) \tau}{\frac{n\pi}{\tau_0}} \frac{T_0}{T_0} 
\]

The line spectrum of the signal \( f_{T_0}(t) \) is shown in Fig. 5.3 for \( \frac{\tau}{T_0} = k^{-1} \). Note that the separation of two consecutive lines in the line spectrum of \( f(t) \) is equal to \( \frac{2\pi}{T_0} \). Hence, as \( T_0 \) gets larger and larger (i.e., as \( f_{T_0}(t) \) becomes less and less periodic-looking) the line spectrum of \( f_{T_0}(t) \) becomes denser and denser. Also, note that the envelope of line spectrum of \( f_{T_0}(t) \) is the Fourier transform (within a constant of proportionality) of the aperiodic pulse of width \( \tau \), centered at \( t = 0 \). It can be shown that most of the
power of \( f_{T_0}(t) \) is located within the frequency interval \([-\frac{2\pi}{\tau}, \frac{2\pi}{\tau}]\). This indicates to us that as we make the signal faster in the time domain (by decreasing the duration \( \tau \) of the pulses) the frequency content of the signal gets expanded in the frequency domain. This statement is a manifestation of what is called in communication theory the “time-bandwidth” product. It can be shown that the time-bandwidth product is constant; that is if we expand the signal in the time domain, we shrink the frequency content of the signal in the frequency domain, and vice versa.

5.3.4 Fourier Transform Properties

The Fourier transform of an aperiodic signal is as useful as the Fourier series of a periodic signal. As a reminder, the Fourier series of a periodic signal allows us to compute the output of a LTI system due to a periodic input, without having to compute the convolution. Furthermore, the Fourier series of a periodic signal provides us with information pertaining to the frequency content of the signal. Similarly, the Fourier transform of a signal allows to compute the output of a LTI system due to an input, without having to compute the convolution. Also, the Fourier transform of the signal contains information about the frequency content of the signal. Equations (5.12) and (5.13) or equations (5.15) and (5.16) are rarely used for the computation of the Fourier transform of signals. Normally, the Fourier transform of a signal is computed by utilizing Fourier transforms of well known signals (see Table 5.1) and Fourier Transform Properties (see Table 5.2).

- The **Linearity Property** says that the Fourier Transform (FT) is a linear operation and if we know the FT of two signals we can easily compute the FT of any linear combination of these two signals.
- The **Scaling Property** says that by contracting (expanding) the signal in the time domain (through multiplication with an appropriate time constant) results in the expansion (contraction) of the FT of the signal (through multiplication with the inverse of the constant); this is another ramification of the “time-bandwidth” product principle. Note that contraction (expansion) of the signal in the time domain corresponds to making the signal faster (slower).
- The **Delay Property** illustrates something that you might have suspected all along. Delaying a signal in the time domain does not change the frequency content of the signal. The frequency content of the signal is determined by looking at the magnitude of the FT of the signal. The delay property reiterates that a time shift will produce a phase shift in the FT of the signal but it will leave the magnitude of the FT unaltered.
- The **Convolution Property** assures us that we do not need to perform convolution to produce the output of a LTI system due to an arbitrary input; simply multiplying the FT of the input and the FT of the impulse response of the system allows us to compute the FT of the output of the system.
- The **Time Differentiation** and **Time Integration** properties demonstrate that making the signal faster/slower (differentiate/integrate) in the time domain results in an expanded/contracted frequency spectrum (FT).
• The **Duality** Property provides us with a new FT pair, every time a FT pair is computed. In particular, if \( F \) is the FT of \( f \), the duality property says that \( F \) would have a FT that is proportional to the mirror image of \( f \) with respect to the \( \omega = 0 \) axis.

• Finally, the **Amplitude Modulation** property tells us that we can shift the frequency content of a signal at an arbitrary frequency by multiplying the signal with a sinusoid. AM systems are based on this property of the FT.

### 5.3.5 Introduction to Communication Systems

The primary function of most communication systems is to transmit information from point \( A \) to point \( B \). Consider, for example, the situation where we have to transmit human voice from point \( A \) to point \( B \) and the channel between points \( A \) and \( B \) is the air. The human voice at point \( A \) is first converted, via a microphone, to an electrical signal. Then, we have to transform this electrical signal into an electromagnetic wave and send it to destination \( B \). The way to achieve that is by using an antenna. Knowing though that the maximum frequency content of a voice signal is around 15 kHz we come to the conclusion that the antenna required must be many miles long! A reasonable size antenna could adapt the signal to the channel only if the signal’s lower frequency is higher than 800 KHz. Hence, in our case, where a voice signal needs to be transmitted using a reasonable-size antenna it suffices to raise the frequency content of the voice signal to the vicinity of 800 KHz. To accomplish the aforementioned goal we ought to multiply the voice signal with a sinusoidal signal. The result of this multiplication is to latch our (the voice signal) \( f(t) \) on the amplitude of the sinusoid, or to modulate the original amplitude of the sinusoid with \( f(t) \). This procedure gives rise to a class of systems that are called amplitude modulation systems.

We distinguish four major classes of amplitude modulation systems:

- Double Sideband Suppressed Carrier Systems (DSB-SC)
- Single Sideband Suppressed Carrier Systems (SSB-SC)
- Double Sideband Large Carrier Systems (DSB- LC)
- Vestigial Sideband Suppressed Carrier Systems (VSB-SC)

### 5.3.6 Double Sideband Suppressed Carrier System (DSB-SC)

Basically **double sideband suppressed carrier (DSB-SC)** modulation arises by multiplying the information signal \( f(t) \) and the carrier wave \( c(t) \), as follows:

\[
m(t) = c(t)f(t) = A_c \cos(\omega_c t)f(t)
\]  

(5.21)

Consequently, the modulated signal \( m(t) \) undergoes a phase reversal whenever the formation signal \( f(t) \) crosses zero as indicated in Fig. 5.4. The envelope of a DSB-SC modulated signal is therefore different from the information signal.

The Fourier transform of \( m(t) \) can easily be determined from the Fourier transform properties of Table 5.2. That is,

\[
M(\omega) = \frac{1}{2} A_c \left[ F(\omega - \omega_c) + f(\omega + \omega_c) \right]
\]  

(5.22)

For the case when the baseband signal \( f(t) \) is bandlimited to the interval \([-\omega_m, \omega_m]\), as in Fig. 5.5(a), we thus find that the spectrum of \( m(t) \) is as illustrated in Fig. 5.5(b). Except for a change in scale factor,
Fig. 5.4: The information signal $f(t)$ and modulated signal $m(t)$

Fig. 5.5: (a) Spectrum of baseband signal. (b) Spectrum of DSB-SC modulated wave
the modulation process simply translates the spectrum of the baseband signal by \( \pm \omega_c \). Note that the transmission bandwidth required by DSB-SC modulation is equal to \( 2\omega_m \), where \( \omega_m \) stands for the maximum frequency content of the signal \( f(t) \).

The baseband signal \( f(t) \) can be uniquely recovered from a DSB-SC wave \( m(t) \) by first multiplying \( m(t) \) with a locally generated sinusoidal wave and then low pass filtering the product, as in Fig. 5.6. It is assumed that the local oscillator signal is exactly coherent or synchronized, in both frequency and phase, with the carrier wave \( c(t) \), used in the product modulator to generate \( f(t) \). This method of demodulation is known as **coherent detection** or **synchronous demodulation**.

It is instructive to derive coherent detection as a special case of the more general demodulation process using a local oscillator signal of the same frequency but arbitrary phase difference \( \phi \), measured with respect to the carrier wave \( c(t) \). Thus, denoting the local oscillator signal by \( A_c' \cos(\omega_c t + \phi) \) and using equation (5.1) for the DSB-SC wave, we find that the product modulator output in Fig. 5.7 is

\[
e(t) = A_c' \cos(\omega_c t + \phi) m(t)
\]

\[
= A_c A_c' \cos(\omega_c t + \phi) f(t)
\]

\[
= \frac{1}{2} A_c A_c' \cos(2\omega_c t + \phi) f(t) + \frac{1}{2} A_c A_c' \cos(\phi) f(t)
\]

The first term in equation (5.23) represents the DSB-SC modulated signal with a carrier frequency of \( 2\omega_c \), whereas the second term is proportional to the baseband signal \( f(t) \). This is further illustrated by the spectrum (Fourier Transform) \( E(\omega) \) shown in Fig. 5.7, where it is assumed that the baseband signal \( f(t) \) is limited to the interval \([ -\omega_m, \omega_m ]\). It is therefore apparent that the first term in equation (5.23) is removed by the low-pass filter in Fig. 5.6, provided that the cut-off frequency of this filter is greater than \( \omega_m \) but less than \( 2\omega_c - \omega_m \). This is satisfied by choosing \( \omega_c > \omega_m \). At the filter output we then obtain a signal given by

\[
e(t) = \frac{1}{2} A_c A_c' \cos(2\omega_c t + \phi) f(t)
\]
5.3.8 The Single Sideband Suppressed Carrier System (SSB-SC)

To maintain this synchronization, we may use a Costas receiver. Another commonly used method is to send a pilot signal at the receiver the pilot signal is extracted by means of a suitably tuned circuit and then translated to the correct frequency for use in the coherent detector.

A block diagram of the quadrature-carrier multiplexing system is shown in Fig. 5.8.

The transmitter part of the system shown in Fig. 5.8(a), involves the use of two separate product modulators that are supplied with two carrier waves of the same frequency but differing in phase by $-90$ degrees. The transmitted signal $m(t)$ consists of the sum of these two product modulator outputs, as shown by

\[ m(t) = A_c f_1(t) \cos(\omega_c t) + A_c f_2(t) \sin(\omega_c t) \tag{5.25} \]

where $f_1(t)$ and $f_2(t)$ denote the two different information signals applied to the product modulators. Thus $m(t)$ occupies a channel bandwidth of $2\omega_c$ centered at the carrier frequency $\omega_c$, where $\omega_m$ is the maximum frequency content of the signals $f_1(t)$ and $f_2(t)$.

The receiver part of the system is shown in Fig. 5.8(b). The multiplexed signal $m(t)$ is applied simultaneously to two separate coherent detectors that are supplied with two local carriers of the same-frequency but differing in phase by $-90$ degrees. The output of the top detector is $\frac{1}{2} A_c A'_c f_1(t)$, whereas the output of the bottom detector is $\frac{1}{2} A_c A'_c f_2(t)$. For the system to operate satisfactorily it is important to maintain the correct phase and frequency relationships between the local oscillators used in the transmitter and receiver parts of the system.

To maintain this synchronization, we may use a Costas receiver. Another commonly used method is to send a pilot signal outside the passband of the modulated signal. In the latter method, the pilot signal typically consists of a low-power sinusoidal tone whose frequency and phase are related to the carrier wave $c(t)$; at the receiver the pilot signal is extracted by means of a suitably tuned circuit and then translated to the correct frequency for use in the coherent detector.

5.3.8 The Single Sideband Suppressed Carrier System (SSB-SC)

The DSB-SC system discussed so far sends the signal $m(t)$ over the channel with Fourier transform shown in Fig. 5.9. One important observation regarding the Fourier transform of $m(t)$ is that it consists of two components...
Fig. 5.8: Quadrature-carrier multiplexing system. (a) Transmitter. (b) Receiver.
sidebands, the upper sideband and the lower sideband. There is no need to send both sidebands over the channel, since the FT of the information signal can be constructed from either the upper or the lower sideband. The single sideband suppressed carrier (SSB-SC), that we intend to discuss in the following, takes advantage of the above observation by sending only one of the sidebands over the channel.

Hence, the SSB-SC system differs from the DSB-SC system because just before the modulated wave \( m(t) \) is ready to head-off for the channel, one of the sidebands is chopped off. The block diagram of an SSB-SC system is shown in Fig. 5.10. Furthermore, the sequence of Fourier transform plots shown in Fig. 5.11 (upper sideband) demonstrate that the system works. The only problem with the block diagram of the system shown in Fig. 5.10 is that it requires the utilization of a "perfect filter" that eliminates completely the frequencies a little bit below \( \omega_c \) and passes without distortion the frequencies a little bit above \( \omega_c \) (we are referring only to the positive frequencies of the upper sideband signal). To relax the requirement for such a "perfect" filter we will consider another way of generating the upper sideband version of the signal \( m(t) \).

For the sake of simplicity consider first the case where the information signal is sinusoidal (i.e., \( f(t) = \cos(\omega_m t) \)). Then, the Fourier transform of \( m(t) = \cos(\omega_m t)\cos(\omega_c t) \) is given in Fig. 5.12. From Fig. 5.12, it is easy to identify the upper sideband of \( m(t) \) in the frequency domain; also, the time-domain representation of this upper sideband is readily defined as follows:

\[
m_+ (t) = \frac{1}{2} \cos(\omega_c + \omega_m) t
\]  

(5.26)

or

\[
m_+ (t) = \frac{1}{2} \cos(\omega_c t)\cos(\omega_m t) - \frac{1}{2} \sin(\omega_c t)\sin(\omega_m t)
\]  

(5.27)

In Fig. 5.13 we depict a way of constructing the upper sideband signal of equation (5.27) without having to resort to the design of perfect filters. In Fig. 5.13 there is a system designated by a block with the \(-90\) notation inside it. We know what this block does to an input signal that is sinusoidal; it shifts the phase of the sinusoidal signal by \(-90\) degrees. We want to be able to characterize this block in the frequency domain.
Fig. 5.10: Block diagram of a SSB-SC system

Fig. 5.11: FT’s at various stage of the SSB-SC system of Fig. 5.10
To do so let us identify the Fourier transforms of an input signal to this block, equals to $\cos(\omega_m t)$ and the output signal produced by this block, equals to $\cos(\omega_m t - 90) = \sin(\omega_m t)$. Obviously,

$$\text{FT}[\sin(\omega_m t)] = -j \pi \delta(\omega - \omega_m) + j \pi \delta(\omega + \omega_m)$$  \hspace{1cm} (5.28)

$$\text{FT}[\cos(\omega_m t)] = \pi \delta(\omega - \omega_m) + \pi \delta(\omega + \omega_m)$$  \hspace{1cm} (5.29)

Based on the above equations it is straightforward to derive the result that

$$\text{FT}[\sin(\omega_m t)] = -j \text{signum}(\omega) \text{FT}[\cos(\omega_m t)]$$  \hspace{1cm} (5.30)

where

$$\text{signum}(\omega) = \begin{cases} 
1 & \omega \geq 0 \\
-1 & \omega < 0 
\end{cases}$$  \hspace{1cm} (5.31)

Equation (5.30) tells us that the transfer function of the block designated by the notation $-90$ is equal to $-j \text{signum}(\omega)$. Now we are ready to generalize the construction of an upper sideband signal using the block diagram of Fig. 5.13 for the case where the information signal $f(t)$ is of arbitrary nature (i.e., not necessarily sinusoidal). In Fig. 5.14, we provide a block diagram for the construction of the upper sideband signal of a modulated wave $m(t)$ when the information signal $f(t)$ is arbitrary. Comparing Fig. 5.13 and Fig. 5.14 we see that they are identical, where in both figures the block designated by the notation $-90$ corresponds to a system with transfer function equal to $-j \text{signum}(\omega)$.

In Fig. 5.15 we show, in a pictorial fashion, why the block diagram of Fig. 5.14 works. What is worth noting is that the output of a $-90$ degree shift system is called the Hilbert transform of the input. Furthermore, if we designate by the input to a $-90$ degree shift system the output (Hilbert transform) of this system is designated by $\hat{f}(t)$. Finally, it is easy to show that

$$\hat{f}(t) = f(t) * \frac{1}{\pi t}$$  \hspace{1cm} (5.32)
Fig. 5.13: Block diagram of a system that generates the upper sideband of $m(t)$ where $f(t) = \cos(\omega_m t)$ and $c(t) = \cos(\omega_c t)$.

Fig. 5.14: Block diagram of a system that generates the upper sideband of $m(t)$.
Fig. 5.15: FT at various stages of the block diagram of Fig. 5.14
5.3.9 Double Sideband Large Carrier (DSB-LC) System

The main idea behind this system is money. Its claim to fame comes from the use of a very cheap receiver called peak or envelope detector. To understand this detector let us first look at the modulated signal \( m(t) \) for two examples of information signals \( f(t) \). In Fig. 5.16 we show \( m_1(t) \) for an information signal \( f_1(t) \) and \( m_2(t) \) for an information signal \( f_2(t) \). The difference between the two information signals \( f_1(t) \) and \( f_2(t) \) is that \( f_1(t) \) is always positive, while \( f_2(t) \) assumes positive and negative values. If we now look at the positive peaks of \( m_1(t) \) we see that a line through them produces the information signal \( f_1(t) \). On the other hand if we look at the positive peaks of \( m_2(t) \) a line through them produces the information signal \( f_2(t) \) only when \( f_2(t) \) is positive and it produces the negative of \( f_2(t) \) when the signal is negative. The reason that the positive peaks of \( m_1(t) \) and \( m_2(t) \) are important is because we can design an inexpensive receiver that traces these positive peaks. Hence, when the signal \( f(t) \) is always positive, this receiver will be able to reproduce the information signal \( f(t) \) from \( m(t) \); this is not the case though when the signal \( m(t) \) assumes positive and negative values. Unfortunately, information signals can be positive or negative. To remedy the problem of a negative information signal we can add a constant to the signal that is larger than or equal to the most negative signal value. Hence, now

\[
m(t) = [A + f(t)] \cos(\omega_c t) = A\cos(\omega_c t) + f(t)\cos(\omega_c t)
\] (5.33)

The Fourier transform of \( m(t) \) is shown in Fig. 5.17 and it is provided by the following expression.

\[
M(\omega) = A\pi[\delta(\omega - \omega_c) + \delta(\omega + \omega_c)] + \frac{1}{2}[F(\omega - \omega_c) + F(\omega + \omega_c)]
\] (5.34)

By looking at Fig. 5.17 we can justify the name DSB-LC for this communication system. The Fourier transform of the modulated wave \( m(t) \) has both of the sidebands (upper and lower). Furthermore, the presence of the pure carrier \( A\cos(\omega_c t) \) is evident by the presence of the two impulses in the Fourier transform of \( m(t) \).

Consider now the special case where the information signal \( f(t) \) is of the form.

\[
f(t) = K\cos(\omega_m t)
\] (5.35)

Based on our previous discussion, the maximum amplitude of the pure carrier that we need to add to the DSB-SC modulated signal to make it a legitimate DSB-LC signal must satisfy the following inequality:
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Fig. 5.17: FT of a DSB-LC signal

\[ A \geq K \]  \hspace{1cm} (5.36)

Then,

\[ m(t) = A(1 + \frac{k}{A}\cos(\omega_m t))\cos(\omega_c t) \]  \hspace{1cm} (5.37)

We usually define

\[ m = \frac{K}{A} \]  \hspace{1cm} (5.38)

the modulation index of the system. Since \( m \)'s maximum value is one and its minimum value is zero, it is often given in percent. In Fig. 5.18 we show the DSB-LC signal for the case of an information signal that is sinusoidal and for various \( m \) values (i.e., \( m = 1 \), \( m < 1 \) and \( m > 1 \)). The case \( m > 1 \) is not allowed because then we are not going to be able to recover the information signal \( f(t) \). In Fig. 5.19 we show a modulated wave for an arbitrary information signal \( f(t) \) and a modulation index \( m < 1 \) or \( m > 1 \).

As we emphasized before, the information signal \( f(t) \) can be recovered from a DSB-LC signal by utilizing a peak (envelope) detector. An envelope detector consists of a diode and a resistor-capacitor (RC) filter (see Fig. 5.20). The operation of this envelope detector is as follows.

On a positive half-cycle of the input signal, the diode is forward biased and the capacitor \( C \) charges up rapidly to the peak value of the input signal. When the input signal falls below this value, the diode becomes reverse-biased and the capacitor \( C \) discharges slowly through the load resistor \( R_l \). The discharging process continues until the next positive half-cycle. When the input signal becomes greater than the voltage across the capacitor, the diode conducts again and the process is repeated. We assume that the diode is ideal, presenting resistance \( r_f \) to current flow in the forward-biased region, and infinite resistance in the reverse-biased region. We further assume that the AM wave applied to the envelope detector is supplied by a voltage source of internal impedance \( R_s \). The charging time constant \( (r_f + R_s)C \) must be short compared with the carrier period \( \frac{2\pi}{\omega_c} \), that is

\[ (r_f + R_s)C \ll \frac{2\pi}{\omega_c} \]  \hspace{1cm} (5.39)
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Fig. 5.18: Effects of varying modulation indices

Fig. 5.19: Importance of sufficient carrier in DSB-LC waveform
Fig. 5.20: Envelope detector. (a) Circuit diagram (b) AM wave input (c) Envelope detector output
so that the capacitor $C$ charges rapidly and thereby follows the applied voltage up to the positive peak when the diode is conducting. On the other hand, the discharging time constant $R_lC$ must be long enough to ensure that the capacitor discharges slowly through the load resistor $R_l$, between positive peaks of the modulated wave, but not so long that the capacitor voltage will not discharge at the maximum rate of change of the modulating wave, that is

$$\frac{2\pi}{\omega_c} \ll R_lC \ll \frac{1}{\omega_m}$$  \hspace{1cm} (5.40)

where $\omega_m$ is the information signal bandwidth. The result is that the capacitor voltage or detector output is nearly the same as the envelope of the AM wave, as shown in Fig. 5.20.

One of the serious flaws of a DSB-LC system is that it wastes transmitter power. To illustrate that consider the DSB-LC signal of equation (5.33). It is easy to show that

$$Power of m(t) = Power of \{A\cos(\omega_c t)\} + Power of \{f(t)\cos(\omega_c t)\} + Power of \{\sqrt{2Af(t)\cos(\omega_c t)}\} \hspace{1cm} (5.41)$$

The power of $A\cos(\omega_c t)$ is equal to $\frac{A^2}{2}$, while the power of $f(t)\cos(\omega_c t)$ can be shown to be equal to the one-half the power of $f(t)$; the latter result is valid under the legitimate assumption that the bandwidth of the information signal is much smaller than the carrier frequency $\omega_c$. Finally, the power of $\sqrt{2Af(t)\cos(\omega_c t)}$ is equal to zero under the assumptions that

(i) the DC value of $f(t)$ is zero.

(ii) the bandwidth of the information signal is much smaller than the carrier frequency.

$$Power of m(t) = \frac{A^2}{2} + \frac{Power of f(t)}{2} \hspace{1cm} (5.42)$$

For the special case where $f(t) = K\cos(\omega_m t)$ we can conclude that

$$Power of m(t) = \frac{A^2}{2} + \frac{A^2m^2}{4} = \frac{A^2}{2}(1 + \frac{m^2}{2}) \hspace{1cm} (5.43)$$

It is worth noting that from the power dedicated to transmit the modulated signal, $\frac{A^2}{2}$ goes to the transmission of the pure carrier and the rest goes to the transmission of the useful signal. If we define the efficiency $\eta$ of our system as the ratio of the power dedicated to the transmission of the useful signal versus the power dedicated to the transmission of the modulated signal we conclude that the efficiency of our DSB-LC system (for a sinusoidal information signal) is equal to

$$Efficiency = \eta = \frac{m^2}{m^2 + 2} \hspace{1cm} (5.44)$$

The best we can do is to let $\eta$ assume its maximum value; this happens when $m = 1$. Then,

$$\eta = 0.33 \hspace{0.1cm} (33\%) \hspace{1cm} (5.45)$$
So 67% of the available transmitter power goes to the carrier (alarming quantity!) Usually, we do not know \( K \) so we might operate at less than 33% efficiency.

The notion of the modulation index can be extended to all signals, as long as they are normalized so that their maximum negative value is unity. To be more specific consider the DSB-LC signal corresponding to an information signal whose maximum negative value is different than unity. Then,

\[
m(t) = A\cos(\omega_c t) + f(t)\cos(\omega_c t)
\]

\[
= A\cos(\omega_c t) + |\min\{f(t)\}| \frac{f(t)}{\min\{f(t)\}} \cos(\omega_c t)
\]

\[
= A[1 + \frac{|\min\{f(t)\}|}{A} f'(t)]\cos(\omega_c t)
\]

(5.46)

where \( f'(t) \) is the normalized version of \( f(t) \). Note that \( f'(t) \) has minimum value of -1. In the above equation we define the modulation index \( m \) to be equal to the ratio

\[
\frac{|\min\{f(t)\}|}{A}
\]

(5.47)

Actually, the aforementioned definition is applicable independently of whether \( f(t) \) has minimum value of unity or not.

5.3.10 Vestigial Sideband Suppressed Carrier (VSB-SC)

The stringent frequency-response requirements on the sideband remover of a SSB-SC system can be relaxed by allowing a part, called vestige, of the unwanted sideband to appear at the output of the modulator. Thus, we simplify the design of the sideband filter at the cost of a modest increase in the channel bandwidth required to transmit the signal. The resulting signal is called vestigial/-sideband suppressed carrier (VSB-SC) system. The suppressed carrier name sterns from the fact that one more time no pure carrier is sent over the channel.

To generate a VSB-SC signal, we begin by generating a DSB-SC signal and passing it through a sideband filter with frequency response \( H(\omega) \) as shown in Fig. 5.21. In the time domain the VSB-SC signal may be expressed as

\[
m(t) = [f(t)\cos(\omega_c t)] * h(t)
\]

(5.48)

where \( h(t) \) is the impulse response of the VSB-SC filter. In the frequency domain the corresponding expression is

\[
M(\omega) = \frac{1}{2}[F(\omega - \omega_c) + F(\omega + \omega_c)]H(\omega)
\]

(5.49)

To determine the frequency-response characteristics of the filter, let us consider the demodulation of the VSB-SC signal \( m(t) \). We multiply \( m(t) \) by the carrier component and pass the result through an ideal lowpass filter, as shown in Fig. 5.22. Thus the product signal is

\[
e(t) = m(t)\cos(\omega_c t)
\]

(5.50)
or, equivalently

\[ E(\omega) = \frac{1}{2}[M(\omega - \omega_c) + M(\omega + \omega_c)] \]  (5.51)

If we substitute for \( M(\omega) \) from equation (5.49) into equation (5.50), we obtain

\[ E(\omega) = \frac{1}{4}[F(\omega - 2\omega_c) + F(\omega)H(\omega - \omega_c) + \frac{1}{4}[F(\omega + 2\omega_c) + F(\omega)]H(\omega + \omega_c)] \]  (5.52)

The lowpass filter rejects the double frequency components and passes only the components in the frequency range \( |\omega| < \omega_m \). Hence the signal spectrum at the output of the ideal lowpass filter is

\[ O(\omega) = \frac{1}{4}F(\omega)[H(\omega - \omega_c) + H(\omega + \omega_c)] \]  (5.53)

We require that the information signal at the output of the lowpass filter be undistorted. Therefore, the VSB-SC filter characteristic must satisfy the condition:

\[ H(\omega - \omega_c) + H(\omega + \omega_c) = \text{constant for } \omega \leq \omega_m \]  (5.54)

The condition is satisfied by a filter that has the frequency response characteristic shown in Fig. 5.23. We note that \( H(\omega) \) selects the upper sideband and a vestige of the lower sideband. It has an odd symmetry about the carrier frequency \( \omega_c \), in the frequency range \( \omega_c - \omega_a < \omega < \omega_c + \omega_a \), where \( \omega_a \) is a conveniently
selected frequency that is some small fraction of $\omega_m$, i.e., $\omega_a << \omega_m$. Thus, we obtain an undistorted version of the transmitted signal. Fig. 5.24 illustrates the frequency response of a VSB filter that selects the lower sideband and a vestige of the upper sideband.

In practice, the VSB filter is designed to have some specified phase characteristics. To avoid distortion of the information signal, the VSB filter should be designed to have a linear phase over its passband $\omega_c - \omega_a < \omega < \omega_c + \omega_a$.

5.3.11 Why modulation?

Modulation is the process by which a property of a parameter of a signal is in proportion to a second signal. The primary reason for using modulation in communication is:

1. To raise up the frequency of a signal to reduce the wavelength such that a relatively small antenna can transmit or receive the signal.
2. To separate base band signals in frequency (or time) so that more than one signal can be transmitted on the same channel.
3. To have the base band signal transformed for ease of transmission.

The degree to which a signal is modulated is measured by a modulation index, which will have a different physical significance for each type of modulation. Every form of modulation has advantages and disadvantages when compared to the others. FM has more strengths than weaknesses compared to the others. Delta Modulation has more weakness than strengths, but all have their applications. Some modulation techniques and their characteristics are listed in Table 5.3.
5.3.12 Aliasing

Aliasing occurs in most forms of modulation when the sampling rate or the carrier frequency is not large enough compared to the maximum message frequency.

Once the spectra have overlapped there is no way to separate the two signals. The Nyquist theorem states that for a sampled signal, such as PAM, the sampling frequency \( f_s \) must be at least twice the maximum frequency of the signal to prevent aliasing from occurring.

5.4 Simulation

In this simulation we are going to see the amplitude modulation and demodulation with different parameters.

Let us consider a sinusoidal message signal of 100 Hz and a sinusoidal carrier in the transmitter of 1000 Hz.

\[
\begin{align*}
    f &= 100; \\
    fc &= 1000; \\
    m &= 0.5; \\
    fs &= 16384;
\end{align*}
\]

MODULATION
1) Construct the time array as following
   \[
   t = [0:1/fs:0.05];
   \]
2) Construct the message signal as
   \[
   \text{signal} = \sin(2 \pi f t);
   \]
3) Calculate the required dc offset from modulation index definition
   \[
   A = \frac{\text{abs(min(signal))}}{m};
   \]
4) Construct the carrier at the transmitter end as
   \[
   \text{carrier1} = \sin(2 \pi fc t);
   \]
5) Modulated output of the transmitter is given by
   \[
   x = (\text{signal}+A) \cdot \text{carrier1};
   \]
   (Notice a dot before the multiplication sign. In Matlab it represents multiplication element by element.)
6) Find the Fourier transform of \text{signal}, \text{carrier1} and \( x \). Name them say \( S1, C1 \) and \( X \) respectively. Use the \texttt{ft} function for this. You have to create your own \texttt{ft} function. The code for this function is given below. Make sure that you saved this function as a separate MATLAB file with the same name \texttt{ft.m} in the Matlab working directory.

```matlab
function [X]=ft(x,fs)
    n2=length(x);
    n= 2^max(nextpow2(fs),nextpow2(n2));
    X1=fft(x,n)/fs;
    X=X1(1:n/8);
```

An example has been given below how you find the Fourier transform using this MATLAB function. You can write the following line of code in your main program to find Fourier transform of a signal.

```matlab
[S1]=ft(signal,fs);
```

For good demodulation of the received signal, the carrier at the receiver end needs to be of the same frequency and phase as the carrier of the transmitter end. In practice this may not be the case.

7) Carrier at the receiver (demodulator) is given by the following equation.

```matlab
carrier2=sin(2 * pi * (fc+delta) * t + phi);
```

Where delta is the frequency error and phi is the phase error. Initially assume that both of them are zero.

8) For demodulation multiply the received signal by the carrier and name it \( x_1 \) and then do a low pass filtering.

For low pass filtering use the following piece of MATLAB code.

```matlab
b=fir1(24,500/fs);
```

(The first parameter gives you the order of the filter. The second parameter is the cutoff frequency. Here it is 200 Hz.)

```matlab
x_demod=conv2(x1,b,'same');
```

Here \( b \) is the coefficient of the filter. So to find the filtered output we did the convolution of \( x_1 \) and \( b \).

9) Find the Fourier transform of \( carrier_2 \) and \( x_{\text{demod}} \). Name them \( C_2 \) and \( S_2 \).

10) Plot the following:

- \( \text{signal} \): message signal before modulation versus time
- \( x \): modulated signal versus time
- \( x_{\text{demod}} \): demodulated signal versus time
- \( S_1 \): message signal in frequency domain
- \( C_1 \): carrier at transmitter end in frequency domain
- \( X \): modulated signal in frequency domain
- \( C_2 \): carrier at the receiver end in frequency domain
- \( S_2 \): demodulated signal in the frequency domain

You may use following MATLAB functions to plot all of them together.

Subplot

Plot

11) Repeat the simulation using square wave carrier at both ends. What happens if you use different wave shapes? Explain. Actually in the hardware experiments that follow you will be using a square wave carrier.

12) Repeat the above simulation with a 'sinc' function as the message signal.

13) Repeat the above simulation with \( \phi = \pi/4, \pi/2 \) in step 7). What happens? Explain.

What happens if delta = 200/400 Hz?

### 5.5 Pre-lab questions

**Non sinusoidal carrier**

Consider a case of double-sideband suppressed carrier amplitude modulation (DSB-SC AM), where instead of having a sinusoid to modulate our signal we use some other periodic signal \( p(t) \) with Fourier series expansion:
\[ p(t) = \sum_{n=-\infty}^{\infty} P_n e^{jn\omega_0 t} \quad (5.55) \]

1. By providing a qualitative example (i.e. assume a specific \( p(t) \), say square wave that we will be using in our lab) show how the spectrum of the modulated signal will look like.

2. How can we demodulate a signal of this kind?

Explain and illustrate all your answers.

## 5.6 Implementation

By this time you already know that in amplitude modulation, the amplitude of the carrier frequency varies with the amplitude of the desired message signal. Typically the carrier will be a cosine or some sine wave but in this lab a square wave is used as the carrier for simplicity in implementation. Amplitude Modulation is implemented by multiplying the message signal with the carrier as in the theory part, which can be expressed as:

\[ m(t) = f(t)c(t) = f(t)\cos(\omega_c t) \quad (For \ a \ cosine \ carrier) \quad (5.56) \]

The end result with sinusoidal modulation is such that the spectrum of \( X(f) \) remains unchanged but is removed in frequency and centered about \( \omega_c \). For any general carrier, the multiplication of two signals in time yields convolution in frequency.

A DSB AM signal is described by the equation:

\[ m(t) = f(t)c(t) = (f(t) + A)\cos(\omega_c t) \quad (For \ a \ cosine \ carrier) \quad (5.57) \]

where \( A \) is the average value of the signal, or the DC component. Typically a message signal will have no DC component.

When \( A > 0 \), the \( m(t) \) is a DSB-LC (Large Carrier) AM signal. In this case the DC term must be added to the message signal. The case when \( A = 0 \) is called Suppressed Carrier (DSB-SC) AM. The modulated index for AM is a measure of how large \( A \) is with respect to the amplitude of \( f(t) \).

\[ m = \frac{|f(t)_{\text{min}}|}{A} \quad (5.58) \]

Where \( f(t)_{\text{min}} \) is the minimum value of the message signal before any DC is added and \( A \) is the amount of DC added. Note that \( m = 100\% \) for \( A = |f(t)_{\text{min}}| \).

There are two forms of demodulation used in this lab:

1. **Synchronous:**
   
   The modulated waveform is multiplied by another carrier of (hopefully) identical frequency and phase to the AM carrier. High frequency terms that result from the multiplication are removed with a filter and only the message is left. Note that the carrier must be regenerated for an actual receiver.
2. Envelop Detection:

When $m < 100\%$, peak detection (sample and hold) can be used. Sample and hold means that the amplitude of the signal at a given point in time is sampled and held until the next sample pulse arrives.

The FET used in the AM modulator and demodulator (Fig. 5.25 and Fig. 5.26) is used as a variable resistor, where the resistance between the drain (D) and the source (S) varies proportional to the voltage applied to the gate (G).

Since there is a square wave varying from 0 to $-V_{cc}$ volts been applied to the gate, the resistance across the source and the drain can be thought as either an open ($R = \infty$) when $V_g = -V_{cc}$ or a short ($R = 0$) for $V_g = 0$. Analysis of the circuit shows a gain of $\frac{V_{out}}{V_{in}} = 1$ for $R = \infty$ (differential amplifier configuration) and a gain of $\frac{V_{out}}{V_{in}} = -1$ when $R = 0$ (inverting amplifier configuration).

5.7 Procedure

5.7.1 AM Modulation

1) Build the modulator shown in Fig 5.25 and Fig. 5.26.

2) Connect the oscilloscope to the function generator output (or the message oscillator output), $V_{in}(t)$ and to the modulator output, $V_{o}(t)$, using direct coupling (DC) throughout. Set $V_{in}(t)$ to a sinewave with $f_m = 100Hz$.

3) Adjust the dc offset and amplitude of the message of the message oscillator to yield a DSB-LC modulated waveform having a modulation index $m = 0.75$. Sketch $V_{in}(t)$ and $V_{o}(t)$. Change $V_{in}(t)$ to a triangular waveform and sketch $V_{in}(t)$ and $V_{o}(t)$. 

Fig. 5.25: AM Modulation

Fig. 5.26: Carrier Generator
For a better view of $V_o(t)$ deactivate the channel that reads $V_{in}(t)$ and set the timebase to 5 ms, i.e., $V_{in}(t)$ can be plotted clearly at timebase equal to 2 ms.

4) Adjust the dc offset of the message oscillator to zero, thus producing DSB-SC. Again sketch $V_{in}(t)$ and $V_o(t)$ first for $V_{in}(t)$ sinusoid and then $V_{in}(t)$ triangular.

5) Obtain once again the DSB-SC signal for $V_{in}(t)$ sinusoidal and 100 = mf Hz. Observe and sketch the spectra of $V_{in}(t)$ and $V_o(t)$, using the spectrum analyzer. To see the modulated output you may adjust the frequency display of the spectrum analyzer (i.e., FFT on oscilloscope) to the value equal to the value of the carrier frequency. That is, use FFT button adjusted to the channel of the $V_o(t)$ reading. The preferable timebase is between 2 ms and 5 ms. The frequency of the modulated signal is between 19 kHz and 24 kHz.

6) Repeat part 5 for DSB-LC. Furthermore, using the spectrum analyzer, determine the modulation index, $m$.

5.7.2 Amplitude Demodulation

The overall general communication system block diagram is given in Fig. 5.27. You have already built the modulator. As a channel, use a small length of wire. There are two types of AM demodulator: the envelope detector (no carrier waveform is required), and the synchronous demodulator (carrier waveform is needed).

5.7.3 Envelope detection

1) Connect the envelope detector of Fig. 5.28.

2) Adjust the oscillator for a DSB-LC modulated waveform using sinusoidal modulation with $f_m = 100$ Hz. Connect $V_d(t)$ to the oscilloscope input.

3) Set C to the minimum value of the capacitor substitution box, noting the output waveform as C is increased. Sketch the optimum demodulated waveform. Compare $V_d(t)$ with the modulating waveform $V_{in}(t)$. Comment.

4) Repeat part 3) for a triangular modulating signal.

5) Now adjust for a DSB-SC waveform, comparing $V_d(t)$ with the modulating waveform $V_{in}(t)$. Does the envelope detector demodulate DSB-SC?
5.7.4 Synchronous detection

1) Connect the synchronous detector of Fig. 5.29.
2) Adjust for a DSB-LC signal. Compare $V_{in}(t)$ with $V_d(t)$. Sketch the waveforms. Has the message been recovered?
3) Adjust for a DSB-SC signal. Again compare $V_{in}(t)$ with $V_d(t)$. Sketch the waveforms. Is it possible to retrieve the original message? Explain.
4) Is it possible to demodulate a DSB-LC signal? A DSB-SC signal?

5.8 Questions and Calculations

1. Why must $m$ be less than 100% for envelope detection?
2. Compare Envelope Detection to Synchronous Detection and list at least one advantage and one disadvantage of each.

N.B. The usual Op-Amp IC DC- voltage trigger is 15 V, some Op-Amp's IC DC- voltage trigger can reach 20V.
### Table 5.1 Some Selected Fourier Transform Pairs

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(\omega) = \mathcal{F}{f(t)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e^{-at}u(t)$</td>
<td>$1/(a + j\omega)$</td>
</tr>
<tr>
<td>$te^{-at}u(t)$</td>
<td>$1/(a + j\omega)^2$</td>
</tr>
<tr>
<td>$e^{-at}$</td>
<td>$\sigma \sqrt{2\pi} e^{-\sigma^2 t^2}$</td>
</tr>
<tr>
<td>$e^{-i\omega t}$</td>
<td>$2/(j\omega)$</td>
</tr>
<tr>
<td>$sgn(t)$</td>
<td>$sgn(\omega)$</td>
</tr>
<tr>
<td>$u(t)$</td>
<td>$\pi \delta(\omega) + 1/(j\omega)$</td>
</tr>
<tr>
<td>$\delta(t)$</td>
<td>$1$</td>
</tr>
<tr>
<td>$1$</td>
<td>$2\pi \delta(\omega)$</td>
</tr>
<tr>
<td>$e^{j\omega_0 t}$</td>
<td>$2\pi \delta(\omega - \omega_0)$</td>
</tr>
<tr>
<td>$\cos \omega_0 t$</td>
<td>$\pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$</td>
</tr>
<tr>
<td>$\sin \omega_0 t$</td>
<td>$-j\pi [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$</td>
</tr>
<tr>
<td>$\text{rect}(t/\tau)$</td>
<td>$\tau \text{Sa}(\omega \tau/2)$</td>
</tr>
<tr>
<td>$\frac{\mathcal{W}}{2\pi} \text{Sa}(\mathcal{W}t/2)$</td>
<td>$\text{rect}(\omega/\mathcal{W})$</td>
</tr>
<tr>
<td>$\frac{\mathcal{W}}{\pi} \text{Sa}(\mathcal{W}t)$</td>
<td>$\text{rect}(\omega/2\mathcal{W})$</td>
</tr>
<tr>
<td>$\Lambda(t/\tau)$</td>
<td>$\tau \text{Sa}(\omega \tau/2)^2$</td>
</tr>
<tr>
<td>$\frac{\mathcal{W}}{2\pi} [\text{Sa}(\mathcal{W}t/2)]^2$</td>
<td>$\Lambda(\omega/\mathcal{W})$</td>
</tr>
<tr>
<td>$\cos(\pi t/\tau)\text{rect}(t/\tau)$</td>
<td>$\frac{2\tau \cos(\omega \tau/2)}{\pi \left(1 - (\omega \tau/\pi)^2\right)}$</td>
</tr>
<tr>
<td>$\frac{2\mathcal{W}}{\pi^2} \frac{\cos(\mathcal{W}t)}{1 - (2\mathcal{W}t/\pi)^2}$</td>
<td>$\cos[\pi \omega/(2\mathcal{W})] \text{rect}[\omega/(2\mathcal{W})]$</td>
</tr>
<tr>
<td>$\delta_\tau(t)$</td>
<td>$\omega_0 \delta_{\omega_0}(\omega)$ where $\omega_0 = 2\pi/T$</td>
</tr>
</tbody>
</table>
### TABLE 5.2 Fourier Transforms Properties

<table>
<thead>
<tr>
<th>Operation</th>
<th>$f(t)$</th>
<th>$F(\omega)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity (Superposition)</td>
<td>$a_1f_1(t) + a_2f_2(t)$</td>
<td>$a_1F_1(\omega) + a_2F_2(\omega)$</td>
</tr>
<tr>
<td>Complex conjugate</td>
<td>$f^*(t)$</td>
<td>$F^*(-\omega)$</td>
</tr>
<tr>
<td>Scaling</td>
<td>$f(at)$</td>
<td>$\frac{1}{</td>
</tr>
<tr>
<td>Delay</td>
<td>$f(t-t_0)$</td>
<td>$e^{-j\omega t_0}F(\omega)$</td>
</tr>
<tr>
<td>Frequency translation</td>
<td>$e^{j\omega t}f(t)$</td>
<td>$F(\omega - \omega_0)$</td>
</tr>
<tr>
<td>Amplitude modulation</td>
<td>$f(t)\cos \omega_0 t$</td>
<td>$\frac{1}{2}F(\omega + \omega_0) + \frac{1}{2}F(\omega - \omega_0)$</td>
</tr>
<tr>
<td>Time convolution</td>
<td>$\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau$</td>
<td>$F_1(\omega)F_2(\omega)$</td>
</tr>
<tr>
<td>Frequency convolution</td>
<td>$f_1(t)f_2(t)$</td>
<td>$\frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(u)F_2(\omega - u)du$</td>
</tr>
<tr>
<td>Duality: time-frequency</td>
<td>$F(t)$</td>
<td>$2\pi F(-\omega)$</td>
</tr>
<tr>
<td>Symmetry: even-odd</td>
<td>$f_e(t)$</td>
<td>$F_e(\omega)$ [real]</td>
</tr>
<tr>
<td></td>
<td>$f_o(t)$</td>
<td>$F_o(\omega)$ [imaginary]</td>
</tr>
<tr>
<td>Time differentiation</td>
<td>$\frac{d}{dt}f(t)$</td>
<td>$j\omega F(\omega)$</td>
</tr>
<tr>
<td>Time integration</td>
<td>$\int_{-\infty}^{\infty} f(t)dt$</td>
<td>$\frac{1}{j \omega} F(\omega) + \pi F(0)\delta(\omega)$, where $F(0) = \int_{-\infty}^{\infty} f(t)dt$</td>
</tr>
<tr>
<td>MODULATION</td>
<td>CHARACTERISTICS</td>
<td></td>
</tr>
<tr>
<td>----------------------------</td>
<td>---------------------------------------------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>1. AMPLITUDE (AM)</td>
<td>• Disregarding noise, original signal can be reproduced exactly at the receiver.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Very susceptible to noise of all kinds.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Can transmit and decode wide band width signals</td>
<td></td>
</tr>
<tr>
<td>2. FREQUENCY (FM)</td>
<td>• Disregarding noise, original signal can be reproduced exactly at receiver.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Not very susceptible to noise.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Limited bandwidth</td>
<td></td>
</tr>
<tr>
<td>3. PULSE WIDTH (PWM)</td>
<td>• Similar to FM except some accuracy in reproducing the message is lost.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Easy to demodulate (LPF)</td>
<td></td>
</tr>
<tr>
<td>4. PULSE AMPLITUDE (PAM)</td>
<td>• Similar to AM except some accuracy is lost.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Can be used in multiplexing</td>
<td></td>
</tr>
<tr>
<td>5. PULSE CODED (PCM)</td>
<td>• Only discrete voltage levels can be encoded.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Repeaters can be used to reduce noise in long distance transmission since the code is binary.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Inefficient use of time because it requires synchronization bits.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Can handle fast slew rates.</td>
<td></td>
</tr>
<tr>
<td>6. DELTA (∆m)</td>
<td>• Does not need synchronization.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Simple, 2-bit code.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>• Cannot accurately encode fast slew rates.</td>
<td></td>
</tr>
<tr>
<td>7. DELTA SIGMA(∆-Σm)</td>
<td>• Same as delta except it can handle fast slew rates.</td>
<td></td>
</tr>
</tbody>
</table>
5.9 Hints & Comments

- When building the demodulation circuit, you need to connect it to the modulation circuit which you should have already built.

- Distinguish between the Ground G, Source S and Drain D nodes of the transistors according to the key note in the lab tool kit to match the circuit schematic provided.

In the modulation part: the modulation index $m$ should be calculated using the following formula.

$$m = \frac{|f(t)_{\text{min}}|}{A}$$

where $f(t)_{\text{min}}$ is the minimum voltage of the input signal and $A$ is the offset shift needed to make the input signal all positive.

In the demodulation part: When using the spectral analysis on the oscilloscope: The modulation index $m$ should be calculated using equation (4.41) where the power of the signal is calculated using its spectrum amplitudes.

$$\text{Power of } m(t) = \frac{A^2}{2} + \frac{A^2m^2}{4} = \frac{A^2}{2}(1 + \frac{m^2}{2})$$ (4.41)

Using the envelope detection or synchronous detection: The role of the capacitors is to sharpen the output of the demodulation so that it matches the input signal, i.e. by calibrating the capacitor values you will make the signal thinner matching the input signal shape and thickness. By increasing the capacitor values further, you will mismatch the original input signal.
6 EXPERIMENT: Frequency Modulation

6.1 Objective
To understand the principles of frequency modulation and demodulation.

6.2 Equipment:
The equipment used in this experiment are:

- Oscilloscope: Rohde & Schwarz RTM 3004
- Function Generator, Tektronix AFG 3022B
- Digital Multimeter, Tektronix DMM 4050
- Triple Output Power Supply, Agilent E3630A
- Op Amp module TL084
- XR2206 VCO
- LM565 PLL

Bring a USB Flash Drive to store your waveforms.

6.3 Theory
6.3.1 Introduction
A sinusoidal carrier \( c(t) = A \cos(\omega c t + \theta_0) \) has three parameters that can be modified (modulated) according to an information signal \( f(t) \).

1. Its amplitude \( A \), which leads us to the class of systems designated as amplitude modulating (AM) systems.
2. Its frequency \( \omega_c \), which leads us to a class of systems designated as frequency modulating (FM) systems.
3. Its phase \( \theta_0 \), which leads us to a class of systems designated as phase modulating (PM) systems.

We have already discussed the class of AM systems. In the sequel we focus on the class of FM and PM systems. Note that we can write that

\[
c(t) = A \cos(\omega_c t + \theta_0) = A \cos(\theta(t))
\]  

(6.1)

where \( \theta(t) \) is often called the angle of the sinusoid. That’s why FM and PM systems are sometimes referred to as angle modulating systems.
6.3.2 Preliminary notions of FM and PM Systems

Consider the carrier \( c(t) = A \cos(\omega_c t + \theta_0) \). We can write

\[
c(t) = A \cos(\theta(t)) \tag{6.2}
\]

where we call \( \theta(t) \) the **instantaneous phase** of the carrier. If we differentiate \( \theta(t) \) with respect to time \( t \) we get a time function that we designate by \( \omega_t \) and we call it the **instantaneous frequency**. That is

\[
\omega(t) = \frac{d\theta}{dt} \tag{6.3}
\]

It is easy to see that the above definition of instantaneous frequency makes sense if we apply it to a pure carrier \( c(t) = A \cos(\omega_c t + \theta_0) \), because then we get

\[
\omega(t) = \omega_c \tag{6.4}
\]

From equation (6.3), above, we see that if you have the instantaneous phase of a sinusoid, you can compute its instantaneous frequency by differentiation. Furthermore, if you know the instantaneous frequency \( \omega_t \) of a sinusoid you can compute its instantaneous phase by integration as follows:

\[
\theta(t) = \int_0^t \omega(\tau) d\tau + \theta_0 \tag{6.5}
\]

Obviously, if we start with \( \omega(t) = \omega_c \) we get

\[
\theta(t) = \omega_c t + \theta_0 \tag{6.6}
\]

Phase and frequency modulations are techniques that modify the instantaneous phase and frequency, respectively, of a sinusoid in a way dictated by an information signal \( f(t) \).

6.3.3 Phase Modulation

Here the information signal \( f(t) \) is placed as a linear term in the instantaneous phase of the carrier. That is

\[
\theta(t) = \omega_c t + \theta_0 + k_p f(t) \tag{6.7}
\]

where \( k_p \) is a constant of the modulating device. Hence, the PM modulated signal is equal to

\[
m_p(t) = A \cos(\omega_c t + \theta_0 + k_p f(t)) \tag{6.8}
\]
6.3.4 Frequency Modulation

Here the information signal gets inserted as a linear term into the instantaneous frequency of the carrier. That is,

\[ \omega(t) = \omega_c + k_f f(t) \]  \hspace{1cm} (6.9)

where \(k_f\) is a constant due to the modulator. In this case the instantaneous phase is equal to

\[ \theta(t) = \omega_c t + \theta_0 + k_f \int_0^t f(\tau) d\tau \]  \hspace{1cm} (6.10)

and as a result, the FM modulated signal looks like

\[ m_f(t) = A \cos(\omega_c t + \theta_0 + k_f \int_0^t f(\tau) d\tau) \]  \hspace{1cm} (6.11)

A plot of FM and PM signals is shown in Fig. 6.1.

![Fig. 6.1: Examples of frequency and phase modulation](image)

Now let us discuss FM and PM simultaneously and get a better insight into their similarities and differences. For simplicity assume that \(\theta_0 = 0\). Then,

\[ m_p(t) = A \cos(\omega_c t + k_p f(t)) \]  \hspace{1cm} (6.12)

and

\[ m_f(t) = A \cos(\omega_c t + k_f \int_0^t f(u) du) \]  \hspace{1cm} (6.13)
Let us take \( m_p(t) \) and find its instantaneous frequency \( \omega(t) \). Indeed,

\[
\omega(t) = \omega_c + k_p \frac{df(t)}{dt}
\]

(6.14)

The above equation tells us that in the PM case the instantaneous frequency has a linear term proportional to the derivative of the information signal \( f(t) \). So, we can say that the PM case is in reality the FM case but with an information signal being the derivative of the actual information signal. In other words, if we have a device that produces FM signals we can make it to produce PM signals by giving it as an input the derivative of the information signal. The above equation also tells us that the FM case is in reality a PM case with the modulating signal being the integral of the information signal. Also, if we have a device that produces PM, we can make it to produce FM by providing to it as an input the integral of the information signal.

Hence, what it boils down to is that we need to discuss either PM or FM and not both. We choose to focus on FM, which is used to transmit baseband analog signals, such as speech or music. PM is primarily used in transmitting digital signals.

Our primary focus in the examination of FM signals will be the analysis of its frequency characteristics. Although it has been a straightforward task to find the Fourier transform of an AM signal the same is not true for FM signals. Let us again consider the general form of an FM signal

\[
m_f(t) = A \cos(\omega_c t) \cos(k_f g(t)) - A \sin(\omega_c t) \sin(k_f g(t))
\]

(6.15)

where \( g(t) = \int_0^t f(u) \, du \). If we are able to find the FT of \( \cos(k_f g(t)) \) and \( \sin(k_f g(t)) \) we can produce the FT of the signal \( m_f(t) \) without a lot of effort. The FT of \( \cos(k_f g(t)) \) and \( \sin(k_f g(t)) \) cannot be found for any \( g(t) \) and in fact it has been found for few \( f(t) \)'s. To get a deeper insight consider \( \cos(k_f g(t)) \) and expand it in terms of its Taylor series.

\[
\cos(k_f g(t)) = 1 - \frac{k_f^2 g^2(t)}{2!} + \frac{k_f^4 g^4(t)}{4!} - \frac{k_f^6 g^6(t)}{6!} + \ldots
\]

(6.16)

If the signal \( f(t) \) is known, its Fourier transform \( F(\omega) \) is also known, and the Fourier transform of \( g(t) \) can be computed. In particular, from well known Fourier transform properties we can deduce that

\[
\begin{align*}
g^2(t) & \rightarrow G(\omega) \ast G(\omega) \\
g^4(t) & \rightarrow G(\omega) \ast G(\omega) \ast G(\omega) \ast G(\omega) \\
& \vdots \rightarrow \vdots
\end{align*}
\]

(6.17)

Based on the above two equations (6.16) and (6.17) we can state that to compute the Fourier transform of \( \cos(k_f g(t)) \) for an arbitrary signal \( g(t) \) becomes a formidable task (we need to compute a lot of convolutions). Furthermore, it seems that the bandwidth of \( \cos(k_f g(t)) \) is infinite (note that every time we multiply a signal with itself in the time-domain we double its bandwidth in the frequency domain). Hence if the bandwidth of \( g(t) \) is \( \omega_m \), the bandwidth of \( g^2(t) \) is \( 2\omega_m \), the bandwidth \( g^4(t) \) is \( 4\omega_m \), and so on. In reality though not all the terms in the Taylor series expansion \( \cos(k_f g(t)) \) contribute equally to the determination of the signal \( \cos(k_f g(t)) \). Notice that in the Taylor series expansion of \( \cos(k_f g(t)) \) the coefficients multiplying the powers of \( g(t) \) get smaller and smaller. This observation will lead us into the conclusion that the bandwidth of the terms \( \cos(k_f g(t)) \) and \( \sin(k_f g(t)) \) is indeed finite, and as a result the bandwidth of the FM signal \( m_f(t) \) is also finite.

To investigate the bandwidth of an FM signal thoroughly we will discriminate two cases of FM signals: the case of **Narrowband FM** and the case of **Wideband FM**.
6.3.5 Narrowband FM

Consider again the FM signal \( m_f(t) \) given by the following equation.

\[
m_f(t) = A \cos(\omega_c t) \cos(k_fg(t)) - A \sin(\omega_c t) \sin(k_fg(t))
\]  

(6.18)

The terms for which FT is difficult to evaluate are: \( \cos(k_fg(t)) \) and \( \sin(k_fg(t)) \). Each one of these terms has a Taylor series expansion involving infinitely many terms. Let us see what happens if each one of these terms is approximated only by their first term in the Taylor series expansion. Then,

\[
\begin{align*}
\cos(k_fg(t)) & \approx 1 \\
\sin(k_fg(t)) & \approx k_fg(t)
\end{align*}
\]  

(6.19)

Obviously, if we make the above substitutions in equation (6.18) we get

\[
m_f(t) = A \cos(\omega_c t) - A k_fg(t) \sin(\omega_c t)
\]  

(6.20)

The advantage of the above equation is that we can evaluate its FT, and consequently the FT of \( m_f(t) \). It is not difficult to see that the bandwidth of \( m_f(t) \) is approximately equal to 2 times the bandwidth of \( f(t) \) (\( f(t) \) is the information signal). Hence, when the above approximations are accurate we are generating an FM signal whose bandwidth is approximately equal to the bandwidth of an AM signal. Since, in most cases an FM signal will occupy much more bandwidth than an AM signal, the aforementioned type of FM signal is called narrowband FM. In the sequel, we are going to identify (quantitatively) conditions under which we can call an FM signal narrowband. These conditions will be a byproduct of our discussion of wideband FM systems.

6.3.6 Wideband FM

To illustrate the ideas of wideband FM let us start with the simplest of cases where the information signal is a single sinusoid. That is,

\[
f(t) = a \cos(\omega_m t)
\]  

(6.21)

Then, the instantaneous frequency of your FM signal takes the form:

\[
\omega(t) = \omega_c + k_f a \cos(\omega_m t)
\]  

(6.22)

Integrating the instantaneous frequency \( \omega(t) \) we obtain the instantaneous phase \( \theta(t) \):

\[
\theta(t) = \omega_c t + \frac{k_f a}{\omega_m} \sin(\omega_m t)
\]  

(6.23)

Consequently, the FM modulated wave is

\[
m_f(t) = A \cos(\omega_c t + \frac{k_f a}{\omega_m} \sin(\omega_m t))
\]  

(6.24)
The quantity \( \frac{k_{fa}}{\omega_m} \) is denoted by \( \beta \) and it is referred to as the modulation index of the FM system. Let us now write the above expression for the signal \( m_f(t) \) in a more expanded form.

\[
m_f(t) = A\cos(\omega_c t)\cos(\beta \sin(\omega_m t)) - A\sin(\omega_c t)\sin(\beta \sin(\omega_m t))
\]  
(6.25)

To simplify our discussion, from now on, we will be referring to the quantity \( \cos(\beta \sin(\omega_m t)) \) as term \( A \) and to the quantity \( \sin(\beta \sin(\omega_m t)) \) as term \( B \). Terms \( A \) and \( B \) are the real and the imaginary part of the following complex exponential function

\[
e^{j\beta \sin(\omega_m t)}
\]  
(6.26)

which is a periodic function with period \( \frac{2\pi}{\omega_m} \). The above function can be expanded as an exponential Fourier series, as follows:

\[
e^{j\beta \sin(\omega_m t)} = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_m t}
\]  
(6.27)

where the coefficients \( C_n \) are calculated by the equation:

\[
C_n = \frac{\omega_m}{2\pi} \int_{-\frac{\pi}{\omega_m}}^{\frac{\pi}{\omega_m}} e^{j(\beta \sin x - nx)} dx
\]  
(6.28)

Let us now make a substitution of variables in the above equation. In particular, let us substitute \( \omega_m t \) with \( x \). Then,

\[
C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(\beta \sin x - nx)} dx
\]  
(6.29)

The above integral cannot be evaluated in closed form, but it has been extensively calculated for various values of \( \beta \)'s and most \( n \)'s of interest. It has a special name, called the \( n \)th order Bessel function of the first kind and argument \( \beta \). This function is commonly denoted by the symbol \( J_n(\beta) \). Therefore, we can write:

\[
C_n = J_n(\beta)
\]  
(6.30)

As a result,

\[
e^{j\beta \sin(\omega_m t)} = \sum_{n=-\infty}^{\infty} J_n(\beta) e^{jn\omega_m t}
\]  
(6.31)

often called the Bessel Jacobi equation. If we evaluate the real and imaginary parts of the right hand side of equation (6.31) we will be able to calculate term \( A \) and term \( B \), respectively. It turns out that if we substitute these values for term \( A \) and term \( B \) in the original equation for \( m_f(t) \) (see equation (6.25)) we will end up with
The property of the Bessel function coefficients (actually Property 1) that led us to the above results is listed below. Some additional properties of the Bessel function coefficients are also listed.

1. $J_n(\beta)$ is real valued.
2. $J_n(\beta) = J_{-n}(\beta)$ for $n$ even.
3. $J_n(\beta) = -J_{-n}(\beta)$ for $n$ odd.
4. $\sum_{n=-\infty}^{\infty} J_n^2(\beta) = 1$

The advantage of equation (6.32) compared to the original equation (6.24) that defined the FM signal $m_f(t)$ is that now, through equation (6.32), we can compute the FT of the signal $m_f(t)$. It will consist of an infinite sequence of impulses located at positions $\omega_c + n\omega_m$ where $n$ is an integer. In reality though, no matter how big $\beta$ is, the significant $J_n(\beta)$’s will be only for indices $n \leq \beta + 1$. Hence, the approximate bandwidth of your FM signal, when the information signal is of sinusoidal nature, is given by the following equation.

$$B = 2(\beta + 1)\omega_m$$

(6.33)

In Fig. 6.2 various plots of the Bessel function coefficients $J_n(\beta)$ are shown. As we can see these plots verify our claim, above, that $J_n(\beta)$ become small for indices $n > \beta + 1$. In Fig. 6.3 we show the FT of signals $m_f(t)$ for various $\beta$ values.

In Table 6.1 the values of the Bessel function coefficients $J_n(\beta)$ are shown for various $\beta$ values. We can use the values of the Table 6.1 to evaluate the bandwidth of the signal $m_f(t)$ as follows. We are still operating under the assumption that the information signal is of sinusoidal nature. As a result, expression (6.32) is a valid representation of our signal $m_f(t)$. Let us now impose the criterion that for the evaluation of the bandwidth of the signal $m_f(t)$ we are going to exclude all terms of the infinite sum with index $n_{\text{max}}$, such that $|J_n(\beta)| < 0.01$ for $n > n_{\text{max}}$. This criterion is often called the 1% criterion for the evaluation of bandwidth. If we find that $n_{\text{max}}$ is the minimum index $n$ that does not violate the 1% criterion then we can claim that the approximate bandwidth of our signal, according to the 1% criterion, is:
Fig. 6.3: Magnitude line spectra for FM waveforms with sinusoidal modulation (a) for constant $\omega_m$ (b) for constant $\Delta \omega$

$$B = 2n_{max} \omega_m$$ (6.34)

It is worth mentioning that the evaluation of bandwidth based on equation (6.33) corresponds to the bandwidth of your FM signal according to a 10% criterion.

The above procedure followed for the evaluation of the FT of $m_f(t)$ can be extended to the cases where the information signal $f(t)$ is a sum of sinusoidal signals, or a periodic signal. In particular, if

$$f(t) = a_1 \cos(\omega_1 t) + a_2 \cos(\omega_2 t)$$ (6.35)

then the phase of our FM signal $m_f(t)$ is provided by the following equation.

$$\theta(t) = \omega_c t + \frac{k_f a_1}{\omega_1} \sin(\omega_1 t) + \frac{k_f a_2}{\omega_2} \sin(\omega_2 t)$$ (6.36)

Omitting the details, we arrive at a representation of the signal $m_f(t)$ such that

$$m_f(t) = A \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} J_n(\beta_1) J_k(\beta_2) \cos((\omega_c + n\omega_1 + k\omega_2) t)$$ (6.37)

where $\beta_1 = \frac{k_f a_1}{\omega_1}$ and $\beta_2 = \frac{k_f a_2}{\omega_2}$. As we can see from the above equation, we now have impulses at $\omega_c \pm n\omega_1, \omega_c \pm k\omega_2$ as well as $\omega_c \pm n\omega_1 \pm k\omega_2$.

Most of the aforementioned discussion regarding the FT and the bandwidth of an FM signal $m_f(t)$ is based on the assumption that the information signal $f(t)$ is a sinusoid, a sum of sinusoids, or a periodic signal. We want to be able to derive a formula for the bandwidth of an FM signal $m_f(t)$ for an arbitrary information signal.
Let us revisit the approximate FM signal bandwidth formula (6.33) derived for a sinusoidal information signal

\[ B = 2(\beta + 1)\omega_m = 2(ak_f + \omega_m) \]  

(6.38)

The second equality in (6.38) is obtained by substituting the value of \( \beta \) with its equal. Now let us pay a closer look at the two terms involved in the evaluation of the approximate bandwidth \( B \). The first term \( ak_f \) is the maximum frequency deviation of the instantaneous frequency \( \omega(t) \) from the carrier frequency \( \omega_c \); it is often denoted by \( \Delta \omega \). The second term \( \omega_m \) is the maximum frequency content of the information signal \( f(t) \). Keeping these two clarifications in mind, we now define the approximate bandwidth of an FM signal \( f(t) \) to be equal to

\[ B = 2(\Delta \omega + \omega_m) \]  

(6.39)

where \( \Delta \omega \) is the maximum frequency deviation from the carrier frequency, and \( \omega_m \) is the maximum frequency content of the information signal \( f(t) \). It is not difficult to show that for an arbitrary information signal \( f(t) \), \( \Delta \omega = k_f \max_t |f(t)| \). Furthermore, to find \( \omega_m \) we first need to compute the FT of the information signal \( f(t) \). Hence, based on the above equation we can claim that the approximate bandwidth of an FM signal is computable even for the case of an arbitrary information signal. It is worth pointing out that the bandwidth formula given above has not been proven to be true for FM signals produced by arbitrary information signals; but it has been verified experimentally in a variety of cases. Equation (6.39) is referred to as Carson's formula (rule) for the evaluation of the bandwidth of an FM signal, and from this point on it can be applied freely, independently of whether the information signal is of sinusoidal nature or not.

One of the ramifications of Carson's rule is that we can increase the bandwidth of an FM signal at will, by increasing the modulation constant \( k_f \), or equivalently, by increasing the peak frequency deviation \( \Delta \omega \). One of the advantages of increasing the bandwidth of the FM signal is that larger bandwidths result in FM signals that exhibit better tolerances to noise. Unfortunately the peak frequency deviation of an FM signal is constrained by other considerations, such as a limited overall bandwidth that needs to be shared by a multitude of FM users. For example, in commercial FM the peak frequency deviation is specified to be equal to 75 KHz.

Let us now say a word about PM. From our previous discussions the form of a PM signal produced by a sinusoidal modulating signal is as follows:

\[ m_p(t) = A\cos(\omega_c t + k_p\omega_m \sin(\omega_m t)) \]  

(6.40)

and the instantaneous frequency is equal to

\[ \omega(t) = \omega_c - k_p\omega_m \sin(\omega_m t) \]  

(6.41)

Hence,

\[ \Delta \omega = ak_p\omega_m \]  

(6.42)

That is \( \Delta \omega \) depends on \( \omega_m \). This is considered a disadvantage compared to commercial FM, where \( \Delta \omega \) is fixed. Carson's rule is also applicable for PM systems but to find the peak frequency deviation of a PM
system you need to find the maximum, with respect to time, of $|\dot{f}(t)|$, where $\dot{f}(t)$ is the time derivative of $f(t)$. Actually, for PM systems

One last comment to conclude our discussion of angle modulation systems. It can be shown that the power of an FM or PM signal of the form

$$m(t) = A\cos(\theta(t))$$  \hspace{1cm} (6.43)

is equal to $\frac{A^2}{2}$.

6.4 Pre-lab Questions

1) Provide a paragraph, where you compare AM and FM modulation (advantages, disadvantages).

2) Why is the use of FM more preferred than PM? Explain your answer. (Hint: Compare the frequency deviations in both cases)

3) Give a short qualitative justification of the fact that FM has more noise immunity than AM.

6.5 Simulation

We are going to do the simulation in Simulink of Matlab. To run the Simulink, enter the 'Simulink' command in the MATLAB command window. It should look as it is shown in the following. Open a new window using the left icon.

Construct the following block diagram for the FM modulation and demodulation simulation. The following blocks will be necessary for your simulation. The paths of the blocks have been given as well.
Just drag and drop the blocks you need for your simulation in your work window. Click the left mouse button, hold and drag to connect the blocks by wire. The parameters for the individual module have been given as an example. To set the parameters of a block just double click on it.

**Signal Generator**
- Wave form: sine
- Amplitude: 1
- Frequency: 1000

**Discrete time VCO**
Phase Lock Loop
Set the simulation stop time at say 0.002 from the simulation -> parameters menu.

Run the simulation from the simulation -> start menu. Then double click on the scopes to see the time domain signals.

Do the simulation with square wave input signal. Is the demodulated signal the faithful reproduction of the input signal? Why?

### 6.6 Implementation

In linear FM (frequency modulation), the instantaneous frequency of the output is linearly dependent on the voltage at the input. Zero volts at the input will yield a sinusoid at center frequency at the output. The equation for a FM signal is:

\[
m_f(t) = \cos(\omega_c t + g(t))
\]  

(6.44)

Where \( \omega_c t + g(t) = \phi_f(t) \) (instantaneous phase) and the instantaneous angular frequency is:

\[
\omega_f(t) = \omega_c t + \frac{dg(t)}{dt}
\]

(6.45)

In linear FM the instantaneous frequency can be approximated by a straight line.

\[
\omega_f(t) = \omega_c + k_f f(t)
\]

(6.46)

Where \( k_f \) is a constant and \( f(t) \) is the input signal. \( \omega_c \) can be positive or negative and \( f_c \) can also vary and is a function of external timing resistor and capacitor values. The relationship between input voltage and output frequency at any given point in time is:

\[
f_{\text{out}} = f_c + k_f V_{\text{in}}
\]

(6.49)

where \( k_f \) is positive for Fig. 6.1 and has units of Hz/Volt. \( \omega_c \)
Let \( f(t) = A \cos(\omega_m t) \), then

\[
m_f(t) = \cos(\omega_c t + \frac{ak_f}{f_m} \sin(\omega_m t))
\] (6.50)

The peak frequency deviation from \( \omega_c \) is \( ak_f 2\pi \) (radian/sec), and the total peak-to-peak deviation is \( 2(ak_f 2\pi) \). The modulation index \( \beta \) for this signal is:

\[
\beta = \frac{ak_f}{f_m}
\] (6.51)

Note that \( \beta \) will vary for each frequency component of the signal.

### 6.6.1 Spectrum of FM

The spectrum of an FM signal is described by Bessel functions. As shown in section 6.3.6, for a single frequency, constant amplitude message, the spectrum of \( m_f(t) \) is:

\[
m_f(t) = \sum_{n=-\infty}^{\infty} J_n(\beta) \cos(\omega_c + n\omega_m) t
\] (6.52)

where \( \frac{ak_f}{f_m} = \beta = \text{FM modulation index} \) and \( J_n(\beta) = \frac{1}{\pi} \int e^{j\beta \sin \theta} - J_n(\beta) d\theta \), which is the \( n^{th} \) order Bessel function evaluated at \( \beta \). Therefore, for each frequency at the input there are infinite number of spectral components at the output with the amplitude or each component determined by the modulation index \( \beta \).

The amplitude of the higher order terms will decrease, on the average, such that there will be a limited bandwidth where most of the energy is concentrated.

### 6.6.2 VCO (FM modulator)

A voltage-controlled oscillator (VCO) converts the voltage at its input to a corresponding frequency at its output. This is accomplished by a variable reactor (varactor) where the reactance varies with the voltage across it. The varactor is part of a timing circuit, which sets the VCO output frequency. There are two limitations for most VCO’s:

1. The input voltage must be small (usually there is an attenuating circuit at the input).
2. The bandwidth is limited for a linear frequency-to-voltage relationship.

### 6.6.3 PLL (FM demodulator)

The phase-locked loop (PLL) is used to demodulate FM. The phases of the input and feedback signals are compared and the PLL works to make the phase difference between the two signals equal to zero.

Fig. 6.4 shows a basic block diagram of a PLL.

The VCO in the feedback is an FM modulator, and the center frequency can be set equal to the center frequency of \( m_f(t) \).

The equation for \( m_f(t) \) is
where \( k_f \int f(\alpha) d\alpha = \theta_1 t \) and the equation of \( e(t) \) is

\[
e(t) = \cos(\omega_c t + k_d \int F(\alpha) d\alpha)
\]  

(6.54)

where \( k_d \int F(\alpha) d\alpha = \theta_2 t \).

Since each signal has the same center frequency \( \omega_c \), the phase comparator compares the instantaneous values of \( \theta_1 \) and \( \theta_2 \). The difference in phase is transformed into a DC voltage level proportional to the phase difference and then amplified to yield \( F(t) \). This voltage is the input for the VCO in the feedback. As a result, the difference between \( \theta_1 \) and \( \theta_2 \) will be made smaller. This process occurs continually, such that \( \theta_1 = \theta_2 \) at all times.

Substituting for \( \theta_1 = \theta_2 \)

\[
k_d \int F(\alpha) d\alpha = k_f \int f(\alpha) d\alpha \quad \text{and} \quad F(t) = \frac{k_f}{k_d} f(t)
\]  

(6.55)

where \( f(t) \) was the original message signal and \( F(t) \) is the demodulated output.

### 6.7 Procedure

#### 6.7.1 Modulation

a. Build the FM modulator shown in Fig. 6.5 (a).

b. Determine the constant \( k_f \) from the following:

1. Use your triple output power supply to apply the input voltages specified in the following table and record the output frequency: (Take screen shots for the modulated signal with its frequency measurement) Adjust the best signal sketch by using the scale knob. You can also use an adequate timebase value.

2. Plot \( f_{out} \) vs. \( V_{in} \). Draw the best straight line through these points. The slope of this line is \( k_f \). Note that \( k_f \) has units of Hz/Volts. What is the measured value of \( k_f \)?

3. According to the XR2206 data sheet, the expected voltage-to-frequency conversion gain is \( k_f = \frac{0.32}{R_1 C_1} \) Hertz/Volts.
Calculate the expected value of $k_f$. Assuming the resistors have a tolerance of $\pm 10\%$ and the capacitors have a tolerance of $\pm 20\%$, how do the measured and calculated values compare?

c. Add input coupling capacitor $C_2$ to the circuit as shown in Fig. 6.5 (b). Set the modulation frequency to $f_m = 2\, KHz$. Fill in the following table using the following equations: (Take screen shots for the modulated signal with its frequency measurement) Adjust the best signal sketch by using the scale knob. You can also use an adequate timebase value.

\[
\Delta f_{\text{peak}} = \beta f_m = \alpha k_f
\]  
\[
|V_{in}| = \frac{\Delta f_{\text{peak}}}{k_f}
\]

d. Use the $V_{in}$ found as the amplitude of the input signal and find $\Delta f_{\text{peak}}$ on the oscilloscope. Compare $\Delta f_{\text{peak}}$ calculated and measured. Display only the FM signal on the oscilloscope and trigger on the rising edge of the waveform such that you see the following Fig. 6.6 (it will look like a ribbon):

This "ribbon" displays all frequencies in the FM signal at once. The minimum and maximum frequencies can be easily detected and directly measured. Recall that $\Delta f_{\text{peak}}$ is only half of the peak-to-peak frequency swing. What parameters determine the bandwidth of an FM signal?

e. View the frequency domain waveform to obtain modulation indices of 2.4, 5.52, 8.65. These are zero carrier amplitude indices. Include calculations to verify your results in your lab report.

### 6.7.2 Demodulation

a. Build the FM demodulator utilizing the LM565N PLL shown in Fig. 6.7. Connect the output of the FM modulator shown in Fig. 6.5 (b) to the input of the FM demodulator.
b. Apply a sinusoidal message signal and observe the demodulated message signal. Sketch both waveforms. How do they compare?

An FM demodulator is typically followed by a filter to block the carrier. Setting the oscilloscope Trigger Coupling to HF REJ will help sync on the modulation frequency.

c. Apply a square wave message signal and observe the demodulated message signal. Sketch both waveforms. How do they compare? Why is the demodulated sinusoidal message more faithfully reproduced than the demodulated square wave?
6.8 Hints & Comments

6.9 Section 6.5 Simulation

In the simulation Matlab Simulink the oscilloscope connected to the input wave might not show the input signal. Be sure to adjust the time instant iteration caliber on oscilloscope.

The output of the two different inputs of the filter, i.e. the sine wave or the square wave will be shown as a sine wave. Although the output of a square wave input through this filter is not necessary a sine wave

The output of the square wave input through the filter is shown as a leveled sine wave (i.e. like a ladder)

6.10 Section 6.7 Procedure

Section 6.7.1 Modulation

In part (b), $V_{in}$ (Volts) is a DC voltage connected from the DC power supply.

The nominal value of $k_f$ is calculated from the resistor and capacitor measured values instead of the nominal (ideal) values.

In part (c), $V_{in}$ (Volts) is an AC voltage fed from the functional generator with input frequency of 2 kHz.

The students should be able to see that the equipment used in frequency demodulation gives a linearized approximation of the varying frequency of the carrier signal (by the input signal amplitude), which is a feature of the demodulation apparatus. This linear approximation is best described by a Bessel function up to a specified order $n$. So, the better the demodulation equipment the more precise is the output signal to the original input signal, and the higher the order of the Bessel function approximating this linearization of frequency. Moreover, the less complicated the input signal used, the better is the demodulation. When simple input signals are used we don’t need a better hardware demodulator. So, simple signals are better demodulated than complex signals when using the same demodulator equipment. The simplicity of the signal is a feature of its spectrum structure. The more harmonics in the spectrum of the input signal the higher is the order of the Bessel function used for correct demodulation (i.e. the more complicated and costly is the demodulator hardware) This is why a sine wave is better retrieved than a square wave by our build circuits, since the spectrum of a sine wave contains one harmonic while that of a square wave contains more harmonics.
Fig. 6.5(a) VCO Determination of modulation constant $K_f$

Fig. 6.5(b) Generation of FM signal
Fig. 6.5: PLL as an FM Demodulator

\[ f_{\text{max}} = \frac{1}{t_{\text{min}}} \]

\[ f_{\text{min}} = \frac{1}{t_{\text{max}}} \]

\[ \Delta f_{\text{peak}} = \frac{f_{\text{max}} - f_{\text{min}}}{2} \]

Fig. 6.6 Frequency components of the modulated signal

Fig. 6.5: PLL as an FM Demodulator
Table 6.1

Bessel Functions of the first kind, $J_n(x)$

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Experiment 1

Author: Author’s Name
Instructor: Professor’s Name

August 20, 2018
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<th>Section</th>
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<td>Cover Page</td>
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<tr>
<td>Objective</td>
<td>A description of the scope and objectives of the experiment</td>
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<td>Equipment</td>
<td>Descriptions of all the equipments used in the experiment</td>
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<td>Prelab</td>
<td>An analysis with the needed details, calculations and proofs</td>
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<td>Conclusion</td>
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<td>(Organization)</td>
<td>This part is not a sectional part of the lab report it is rather a metric used by the instructor to measure:</td>
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1 - **Graphs and Figures:** The figures should be clear with good resolution (not necessarily colored) and should show the considered measurements (mainly from oscilloscope) with a well presented caption and a discussion of the measurements presented.

Provide figures for each measurement you take from the oscilloscope.